Bernstein Bound is Tight

Repairing Luykx-Preneel Optimal Forgeries

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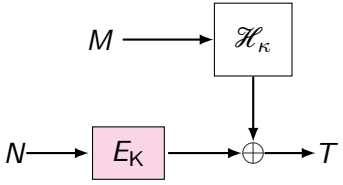
CRYPTO 2018

Wegman-Carter-Shoup (WCS) MAC



- Nonce based Authenticator
- Initial variant (WC authenticator) due to Wegman and Carter [WC81]
- Use of Block cipher E_K due to [Sho96]

Brief History of WC Authenticator

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  - one-time authentication protocol
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  - Issue: a fresh key of size as large as message
- WC authenticator uses strongly universal$_2$ hash function $\mathscr{H}_\kappa$ (based on [CW79]).
  - $R_1, R_2, \ldots,$ is a sequence of secret keys
  - message number n (unique) and a message M
  - Tag: $\mathscr{H}_\kappa(M) \oplus R_n$.

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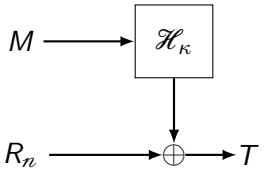

- universal$_2$ is relaxed to a weaker hash AXU in [Kra94/Rog95]
  - $\Pr(\mathscr{H}_\kappa(M) \oplus \mathscr{H}_\kappa(M') = \delta)$ is small
- **polynomial hashing** over n-bits: $\mathrm{Poly}_\kappa(M) := m_d \cdot \kappa \oplus \cdots \oplus m_1 \cdot \kappa^d$ is $\frac{d}{2^n}$-AXU

Getting rid of onetime masking

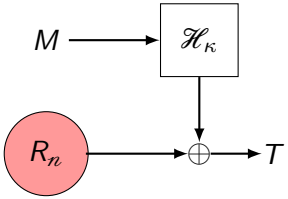

Figure: We can compute R_n directly from n and a secret key.

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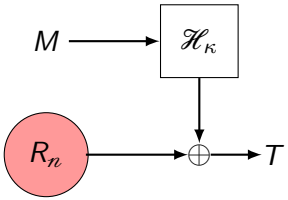

Figure: We can compute R_n directly from n and a secret key.

- Use PRBG (Brassard [Bra83]).
- Sequential in nature.
  - Direct efficient computation of R_n (Blum-Blum-Shub PRBG)
  - also modeled as pseudorandom function.

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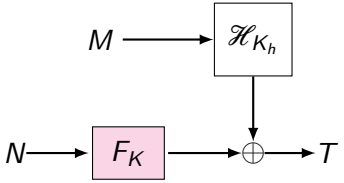
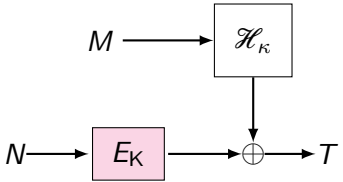


- Use pseudorandom function

Finally - We have WCS



- Use pseudorandom function
- The block cipher (pseudorandom permutation) is widely available. Shoup analyzed WC when PRF is replaced by PRP.

In this Talk

We briefly revisit the security analysis.

- Different attacks.
- Shoup's security guarantee.
- Bernstein's bound and interpretation.

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Recent development on WCS.

- Missing difference Problem [LS18].
- Luykx-Preneel "optimal" forgeries [LP18] using false key set.

Identify the issues of Luykx-Preneel forgeries.

In this Talk (contd.)

We resolve it here.

- We prove the optimality of Bernstein Bound.
- False-key based approach, but different analysis:

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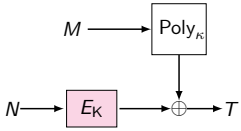
Finally, extend this to show tightness of GCM security

Polynomial Hashing based WCS

Nonce Misuse Forgery



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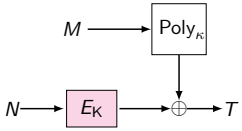

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- nonce misuse (Joux's forbidden attack):
  1. T and T' tags of (N, M) and $(N, M') \Rightarrow$

$$\mathrm{P}_M(\kappa) \oplus \mathrm{P}_{M'}(\kappa) = T \oplus T'$$

  2. solve the hash key (solving polynomial equation).

Polynomial Hashing based WCS

Nonce Respecting Forgery

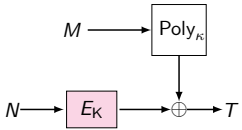

Figure: T is a tag of (N, M).

$$(N, M', T') \text{ is invalid } \kappa \notin \mathrm{Sol}(\mathrm{P}_M(\kappa) \oplus \mathrm{P}_{M'}(\kappa) = T \oplus T').$$

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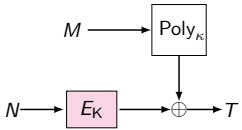

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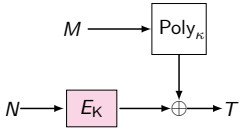

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- d disjoint solutions for each forging attempt.
- success probability after v forging attempts: $v \cdot \epsilon = \frac{v \cdot d}{2^n}$.

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- Bernstein Bound: For all q and v

$$v \cdot \epsilon \cdot (1 - \frac{q}{2^n})^{-\frac{q+1}{2}} \approx v \cdot \epsilon \cdot e^{q^2/2^n}. \tag{3}$$

Interpretation of Shoup's and Bernstein Bound polynomial hash ($\epsilon = d/2^{-n}$) and $v = 1$

Compare: advantage $= \eta$

- Classical bound: $(v + q) \ll 2^{n/2} \Rightarrow \eta$ is small
- Shoup's bound: $q \leq \frac{2^{n/2}}{\sqrt{d}} \Rightarrow \eta \approx 2^{-n}$
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Example: $n = 128$ and $d = 2^{20}$. Data limit is set for advantage 2^{-32}.

- Classical bound: $(v + q) \leq 2^{48.5}$.
- Shoup's bound: $q \leq 2^{54}$.
- Bernstein bound: $q \leq 2^{64}$.

Missing Difference Problem

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Let $\mathscr{L}$, $\mathscr{L}'$ and $\mathscr{S}$ be three lists of n-bit strings satisfying the missing condition:

$$\exists\, s \in \mathscr{S}, \quad s \notin \mathscr{L} \oplus \mathscr{L}'.$$

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Complexity Finding Questions:

1. Let $\mathcal{S} = \{0,1\}^n$. How large the lists should be to ensure the missing condition?
2. How efficiently (both time and memory) we can compute s?

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  3. Coupon collecting problem: Expected number of tries to collect all N coupons (here $2^n - 1$) is $N \log N$

Recovering Hash Key: Approach 1

- For single block message m, tag $T = E_K(N) \oplus \kappa \cdot m$.
- Hash-key recovery algorithm
  1. queries $(N_i, 0)$ and $(N_i', 1)$. Response T_i and T_i' $(1 \leq i \leq q)$
  2. Note, $T_i = E_K(N_i)$ and $T_i' = E_K(N_i') \oplus \kappa$.
  3. So, $\kappa \neq T_i \oplus T_j$ (as $N_i \neq N_j'$).
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  4. κ is the missing number for the sum of the lists $\mathscr{L}$ (of T_i values) and $\mathscr{L}'$ (of T_i' values).
- The assumption on uniformity and independence is wrong.

Luykx-Preneel Forgery: Approach 2

- τ denotes transcript $((N_1, m_1, T_1), \ldots, (N_q, m_q, T_q))$
- Consider False-Key set $\mathscr{F}_\tau$.

$$\mathscr{F}_\tau = \{x : \mathscr{H}_x(M_i) \oplus T_i = \mathscr{H}_x(M_j) \oplus T_j,\ i \neq j\}$$

- $\kappa \notin \mathscr{F}_\tau$ (if not, $E_K(N_i) = E_K(N_j)$) Hope: false-key set almost exhaust the key-space
- Choose an element randomly from $\mathscr{F}_\tau^c$ as a guess of κ
- Key-recovery advantage is at least $\frac{1}{2^n - Ex(|\mathscr{F}_\tau|)}$.

Luykx-Preneel Forgery: Approach 2 (contd.)

Theorem(LP18)

1. $Ex(\mathscr{F}_\tau) \geq q^2/4$ for all $q < 2^{n/2}$.
2. KR advantage is at least $\frac{1}{2^n - q^2/4}$ for all $q < 2^{n/2}$.

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- In the range $q \leq 2^{n/2}$, the random guess shows the optimality.
- For $q \geq 2^{n/2}$, [LP18] did not show anything.

Our Work: Resolving The Issue

- Consider true key approach (complement of false key).
- We have shown KR-advantage is at least

$$\frac{1}{1 + 2^n e^{-\frac{q^2}{2^{n+1}}}}.$$

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- KR is at least $1/2$ for $q = 2^{n/2} \cdot \sqrt{n}$.
- We extend our analysis for GCM.

Analysis for random messages

Hash-key recovery algorithm

1. queries:(N_i, m_i) / Response:T_i – m_i's are chosen randomly
2. $\kappa \in \mathscr{T}_\tau := \{x : \underbrace{T_i \oplus x \cdot m_i}_{R_{i,x}} \neq \underbrace{T_j \oplus x \cdot m_j}_{R_{j,x}} \text{ for all } i \neq j\}$.
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1. for all $x \neq \kappa$, $R_{i,x} := E_K(N_i) \oplus (\kappa \oplus x) \cdot m_i$ – uniform and independent.

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3. $|\mathscr{T}_\tau| = \sum_x I_x$ where I_x is indicator r.v. taking 1 if $R_{i,x}$'s are distinct.

Analysis for random messages

- $\Pr(I_x = 1) = \prod_{i=1}^{q-1}(1 - \frac{i}{2^n}) \approx e^{-\frac{q^2}{2^{n+1}}}$ (birthday paradox bound)

$$\mathbb{E}(|\mathscr{T}_\tau|) = 1 + \sum_{x \neq \kappa} \mathbb{E}(I_x)$$
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- KR advantage is at least $\frac{1}{1+(H-1)e^{-\frac{q^2}{2^{n+1}}}}$.
- Choose $q = 2^{n/2}\sqrt{\log H}$.

Analysis for fixed messages

- $R_{i,x} := E_K(N_i) \oplus (\kappa \oplus x) \cdot m_i$ are no longer independent and uniform.
- Apply WOR distribution of $V_i = E_K(N_i)$

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Lemma

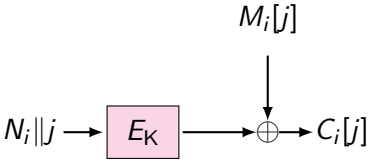
Let $\mathsf{V}_1, \ldots, \mathsf{V}_q$ *be a uniform without replacement sample from* B *of size* N *and* $a_1, \ldots, a_q \in B$ *be some distinct elements, for some* $q \leq N/6$*. Then,*

$$p_x := \Pr(\mathsf{V}_1 + a_1, \ldots, \mathsf{V}_q + a_q \text{ are distinct}) \leq e^{-q^2/4N}.$$

- Apply this lemma with a_i as $(\kappa \oplus x) \cdot m_i$.
- Rest is similar.

Overview of GCM and Its Attack

ciphertext generation



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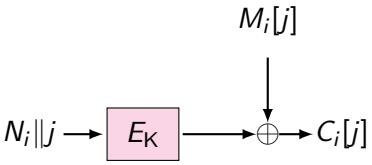
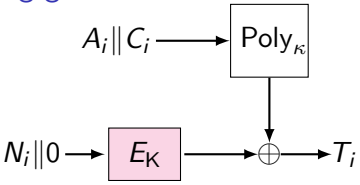


tag generation



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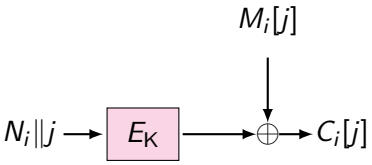

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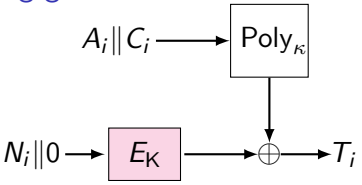

- Encryption query for random messages M_i (same A_i) and let C_i and T_i be responses.
- Construct false key set by comparing $E_K(N_i \| j)$ and $E_K(N_i \| 0)$.
- Hash key recovery whenever ℓq^2 is about $2^n \cdot n$.

More Formally –

- Let M_i be ℓ blocks messages and C_i, T_i be responses.
- True key set $\mathscr{T}_\tau := \{x : \underbrace{T_i \oplus Poly_x(A \| C_i)}_{R_{i,x}} \neq M_i[j] \oplus C_i[j] \text{ for all } i \neq j\}$.
- Clearly, $\kappa \in \mathscr{T}_\tau$.
- As M_i's are random, C_i's are random and using random message analysis, the key recovery advantage is at least

$$\frac{1}{1 + 2^n \cdot e^{\frac{-\ell q^2}{2^n}}}$$

- Query complexity: $\ell q^2 = n \cdot 2^n$.

Conclusion

- Luykx-Preneel forgeries (as it is) is not better than random guess.
- Idea of false key set or missing difference problem is useful.
- We provide correct analysis with higher complexities.
- Applicable for more general set up.
    1. arbitrary hash functions.
    2. applicable for both random and any fixed messages.
- Extend the result for GCM.

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Thank You.