RUB

RUHR-UNIVERSITÄT BOCHUM

Dissection-BKW

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Andre Esser, Felix Heuer, Robert Kübler,
Alexander May, Christian Sohler
Horst Görtz Institute for IT Security
Ruhr University Bochum

hgi
Horst Görtz Institut
für IT-Sicherheit

What is LPN?

Learning Parity with Noise (LPN) Problem

Given: $(\mathbf{a}_i, \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i),\ \mathbf{a}_i \xleftarrow{\$} \mathbb{F}_2^k,\ \Pr[e_i = 1] = \tau < \frac{1}{2}$

Find: $\mathbf{s} \in \mathbb{F}_2^k$

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- Cryptographic applications [HB01, Ale03, HKL$^+$12, DV13]

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- Cryptographic applications [HB01, Ale03, HKL+12, DV13]
- Solve LPN: BKW algorithm [BKW00]
  - Time = **Memory** = **Samples**, slightly subexponential
  - only small experiments [BTV16, EKM17]

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- Cryptographic applications [HB01, Ale03, HKL+12, DV13]
- Solve LPN: BKW algorithm [BKW00]
  - Time = **Memory** = **Samples**, slightly subexponential
  - only small experiments [BTV16, EKM17]
- **Goal:** BKW-variant applicable for any given memory

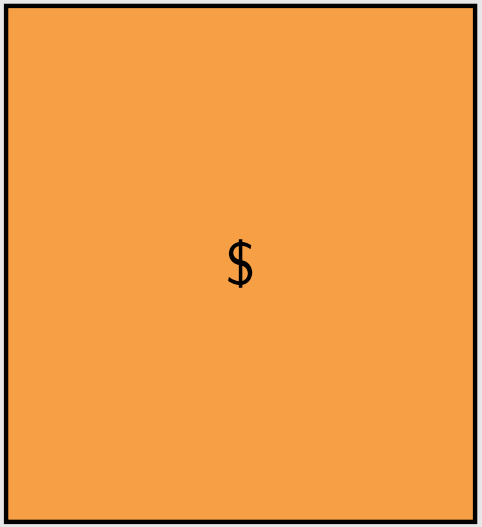
Illustration of “BKW”

$$(\mathbf{a}_1, \langle \mathbf{a}_1, \mathbf{s} \rangle + e_1)$$
$$(\mathbf{a}_2, \langle \mathbf{a}_2, \mathbf{s} \rangle + e_2)$$

Illustration of “BKW”

$$
\begin{aligned}
&(\mathbf{a}_1, \langle \mathbf{a}_1, \mathbf{s} \rangle + e_1) \\
+\ &(\mathbf{a}_2, \langle \mathbf{a}_2, \mathbf{s} \rangle + e_2) \\
\hline
= (&\underbrace{\mathbf{a}_1 + \mathbf{a}_2}_{\mathbf{a}'}, \underbrace{\langle \mathbf{a}_1 + \mathbf{a}_2, \mathbf{s} \rangle}_{\langle \mathbf{a}', \mathbf{s} \rangle} + \underbrace{e_1 + e_2}_{\mathbf{e}'})
\end{aligned}
$$

Illustration of “BKW”



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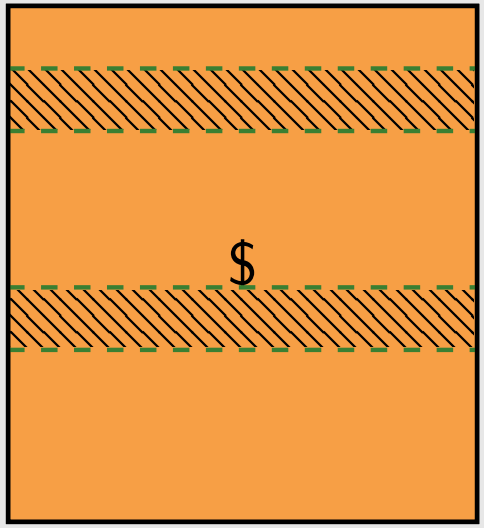

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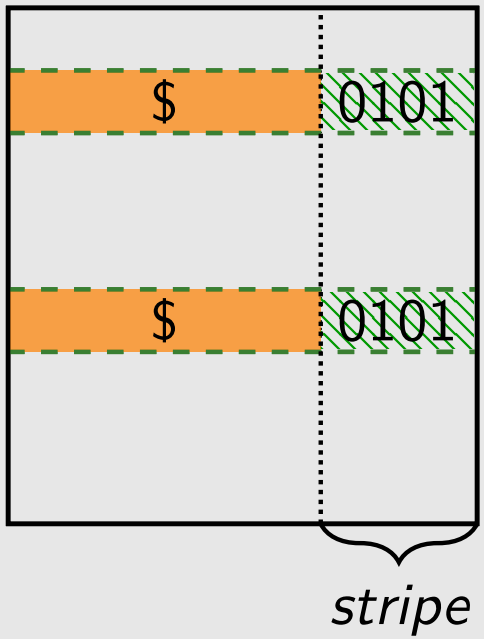

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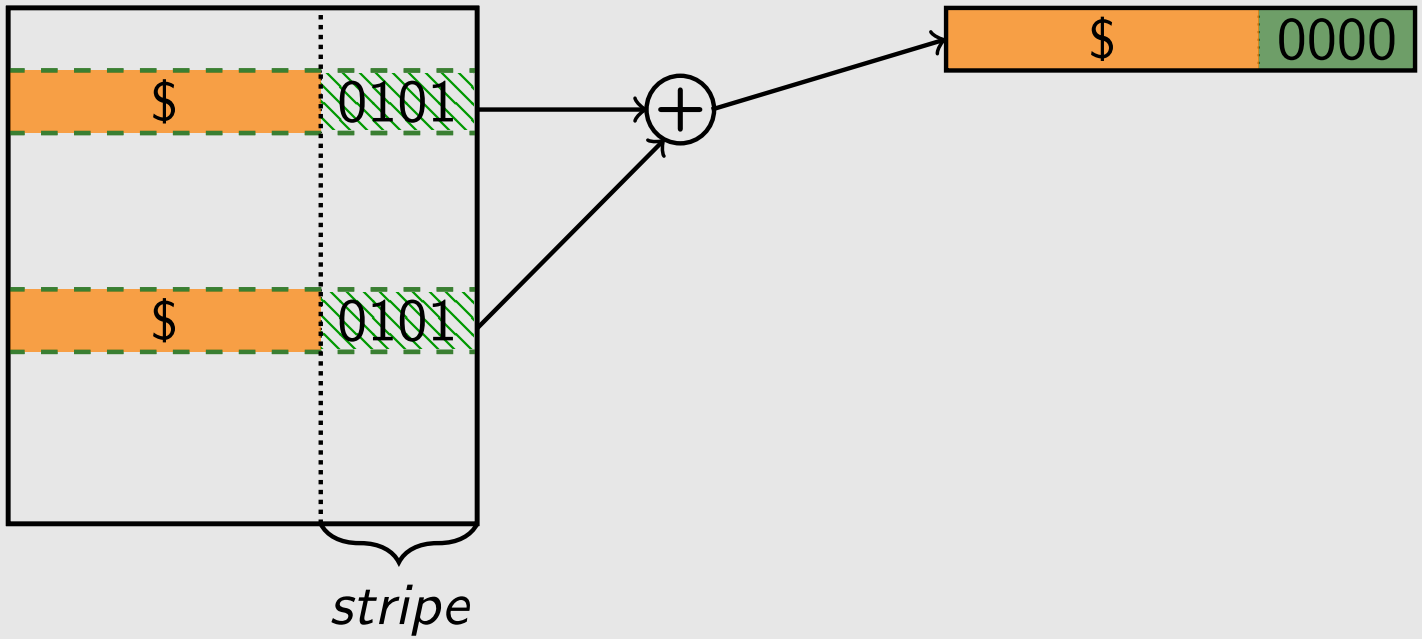

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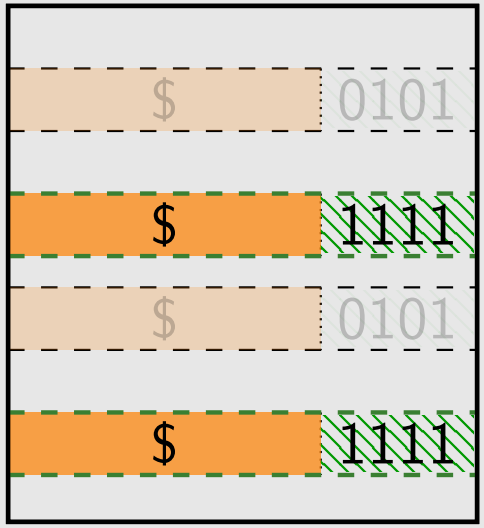

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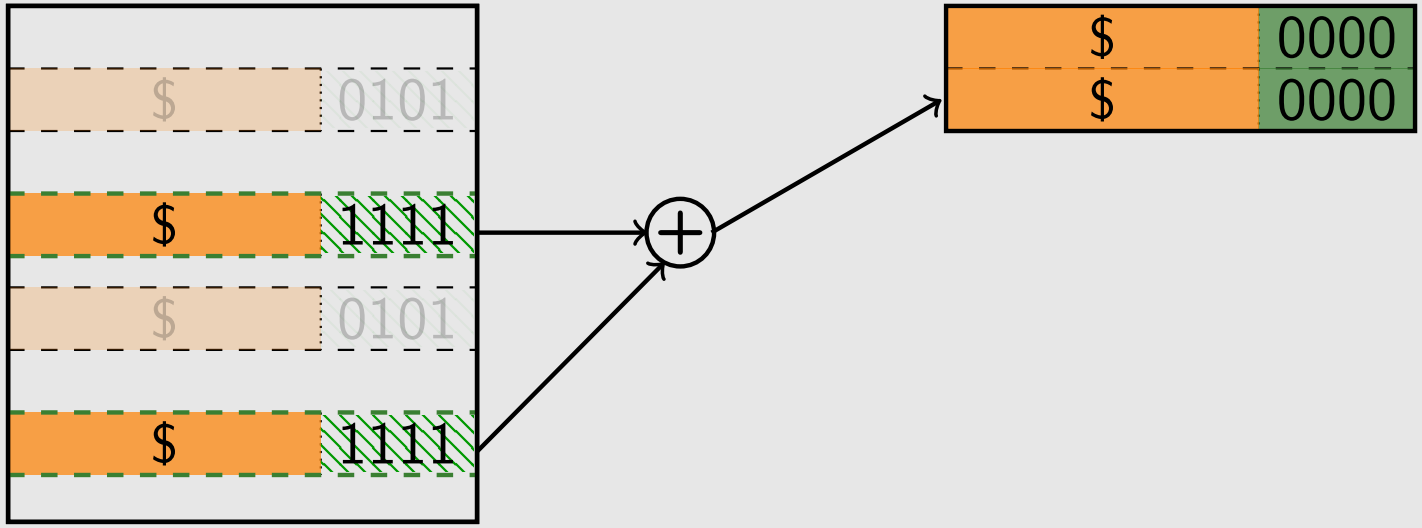

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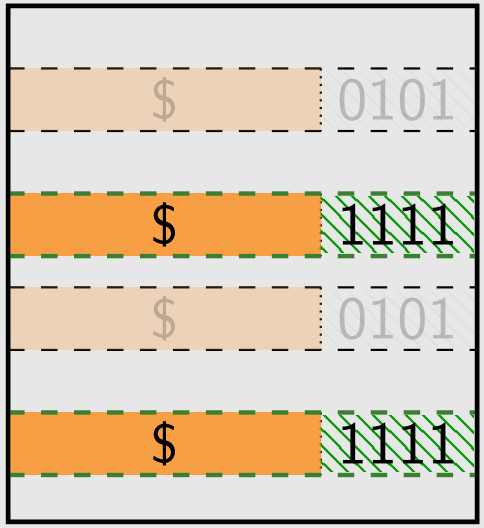

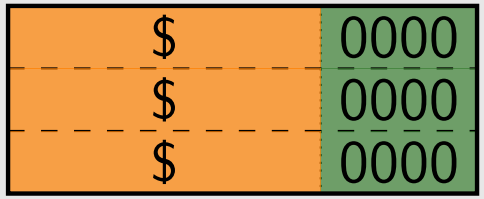

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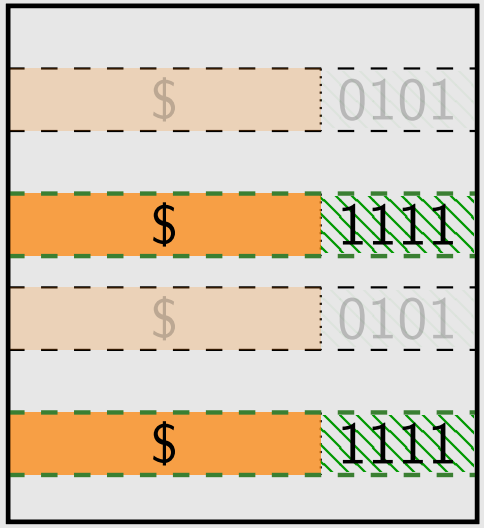

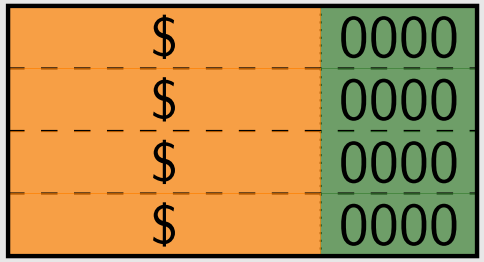

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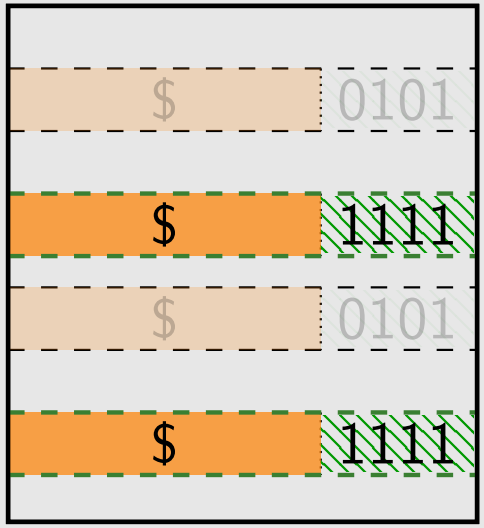

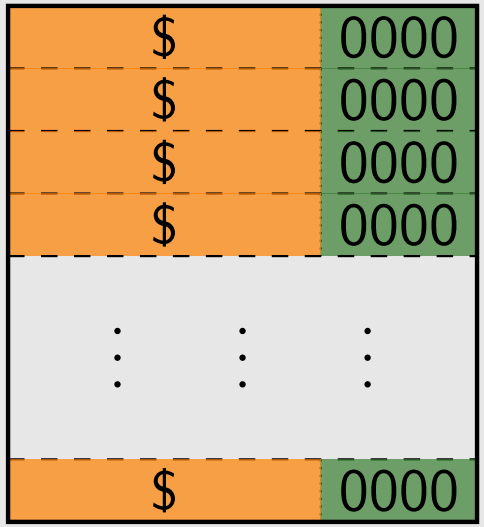

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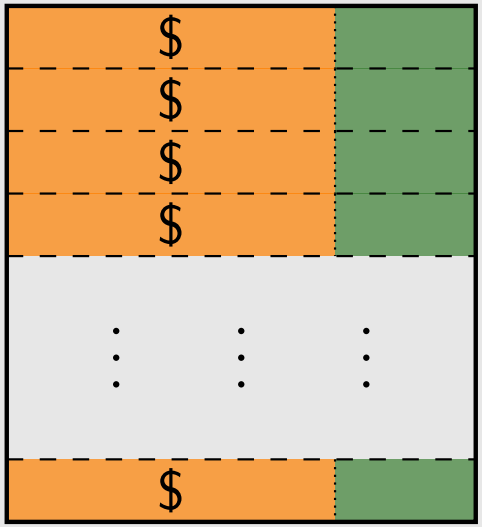

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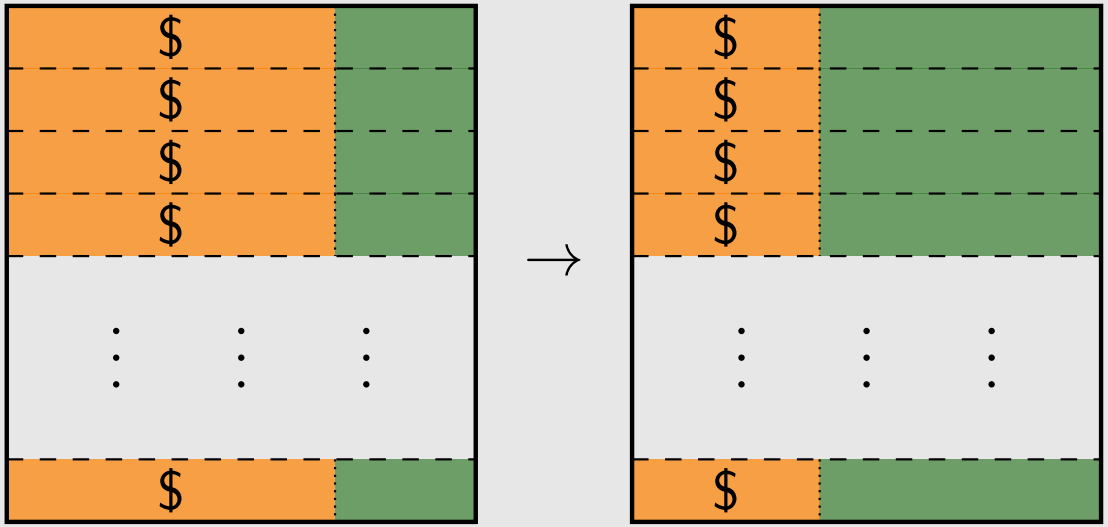

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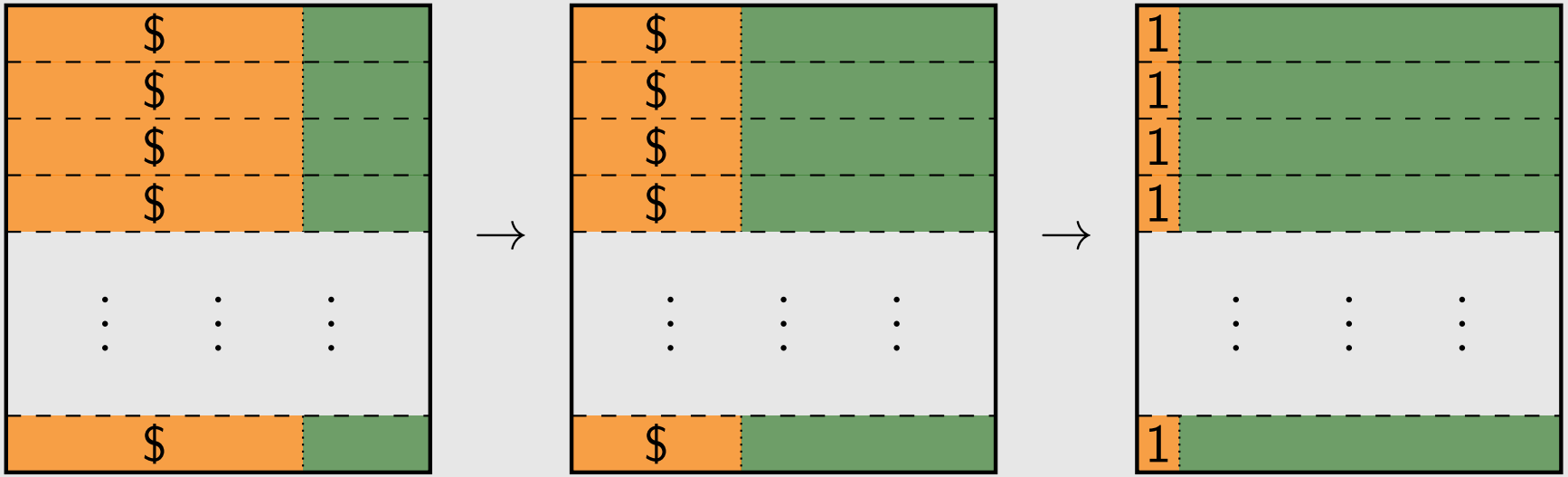

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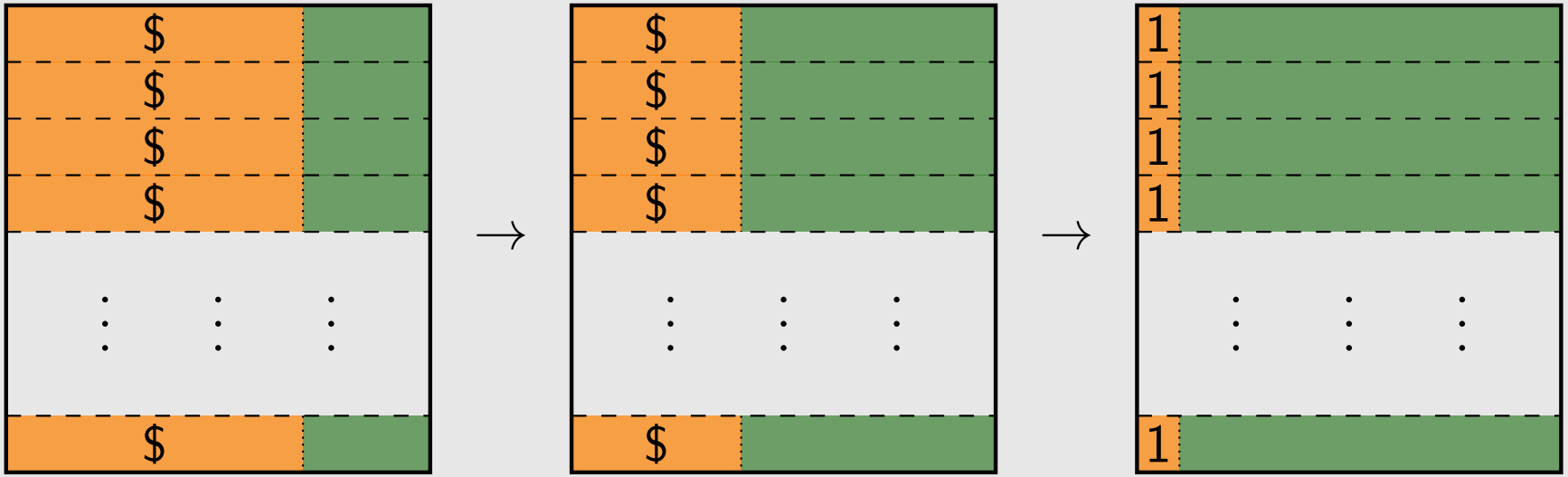
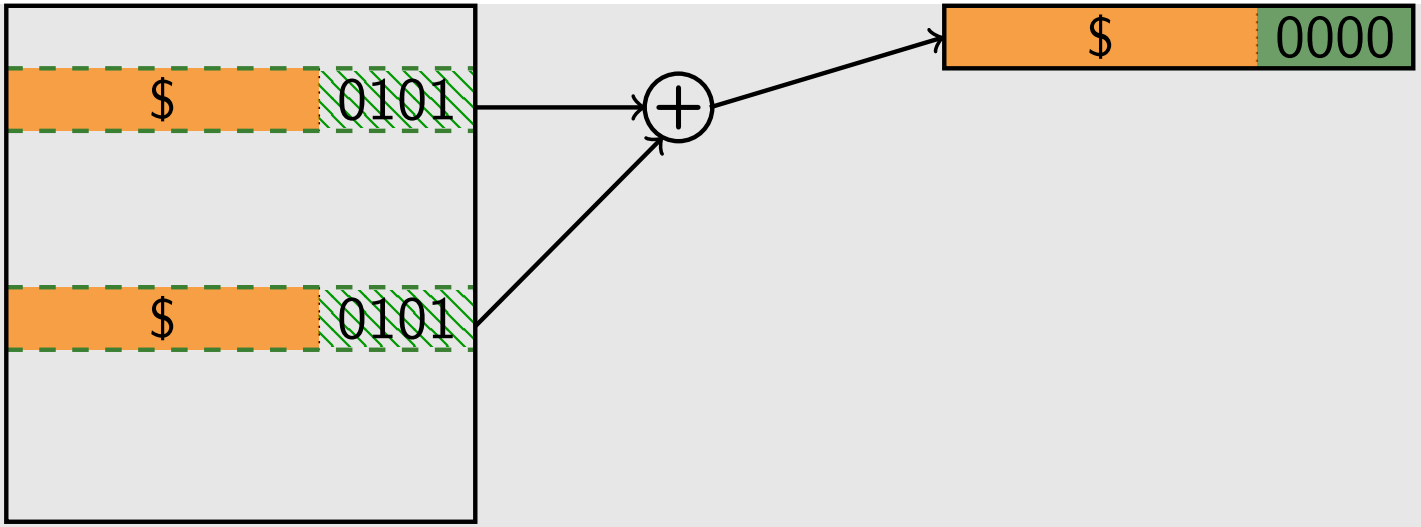


- $\mathbf{a}_i = (1, 0, 0, \ldots, 0) \Rightarrow (\mathbf{a}_i, \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i) = (\mathbf{a}_i, s_1 + e_i)$
- Majority vote!

BKW Theorem [BKW00, LF06]

`BKW` solves LPN in time, memory and sample complexity $2^{k/\log k}$.

c-sum Observation



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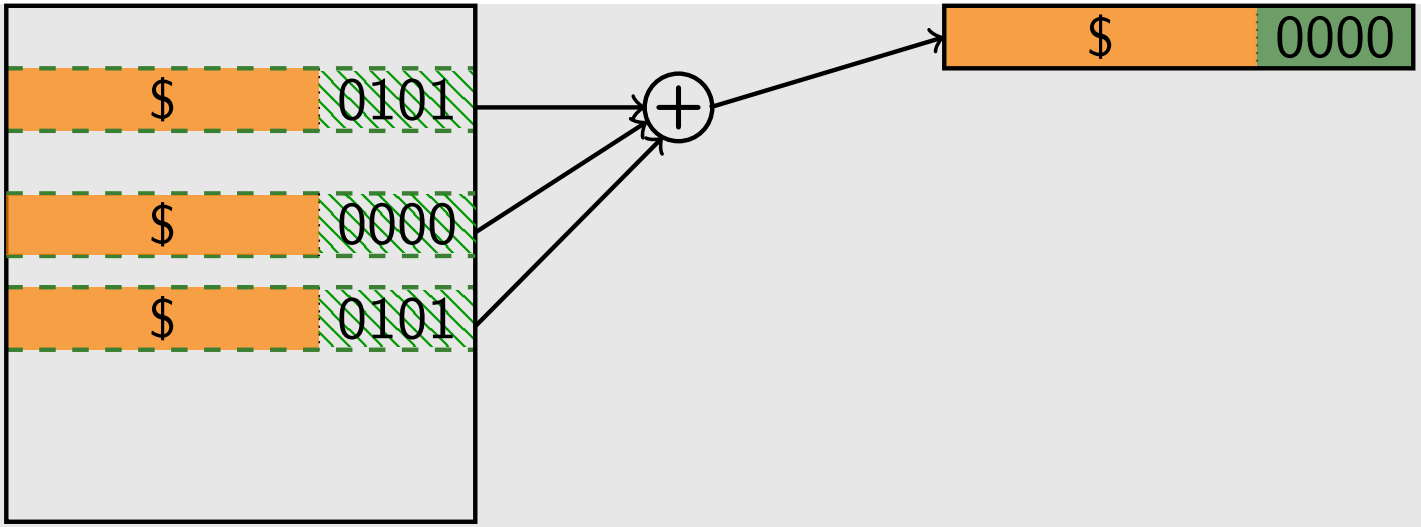

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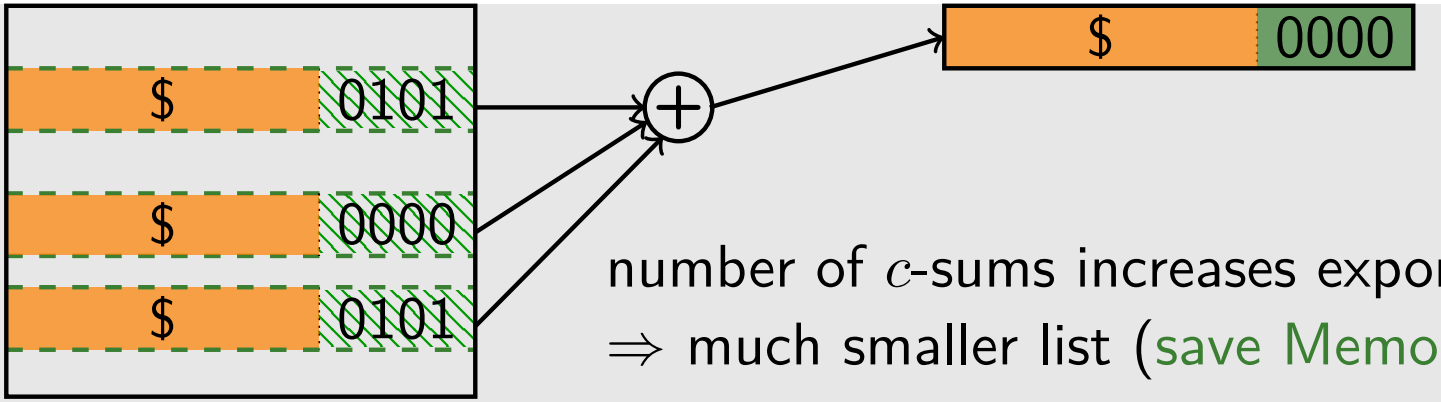

number of c-sums increases exponentially in c

$\Rightarrow$ much smaller list (save Memory & Samples)

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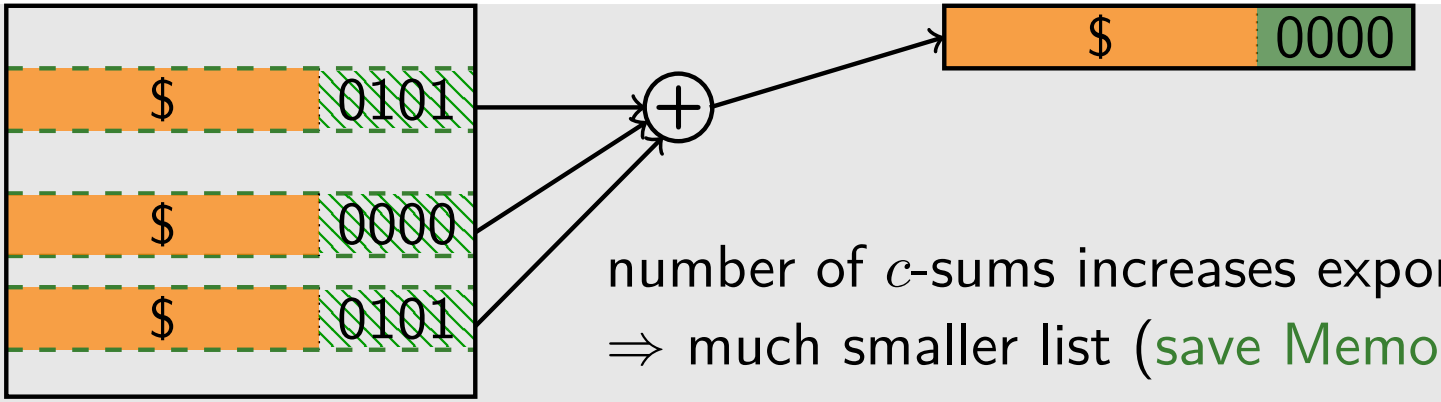

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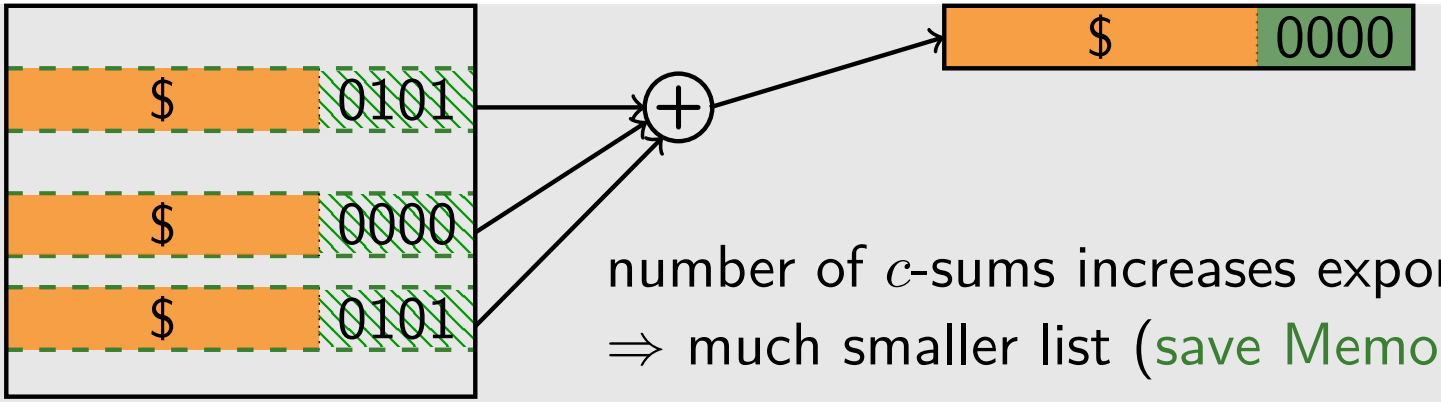
c-sum-Problem (c-SP)

Given a list L of N uniformly distributed elements from $\mathbb{F}_2^b$.
Find N combinations of c elements from L that each add up to 0^b.

c-sum Observation



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$$\binom{N}{c}/2^b \stackrel{!}{\geq} N$$

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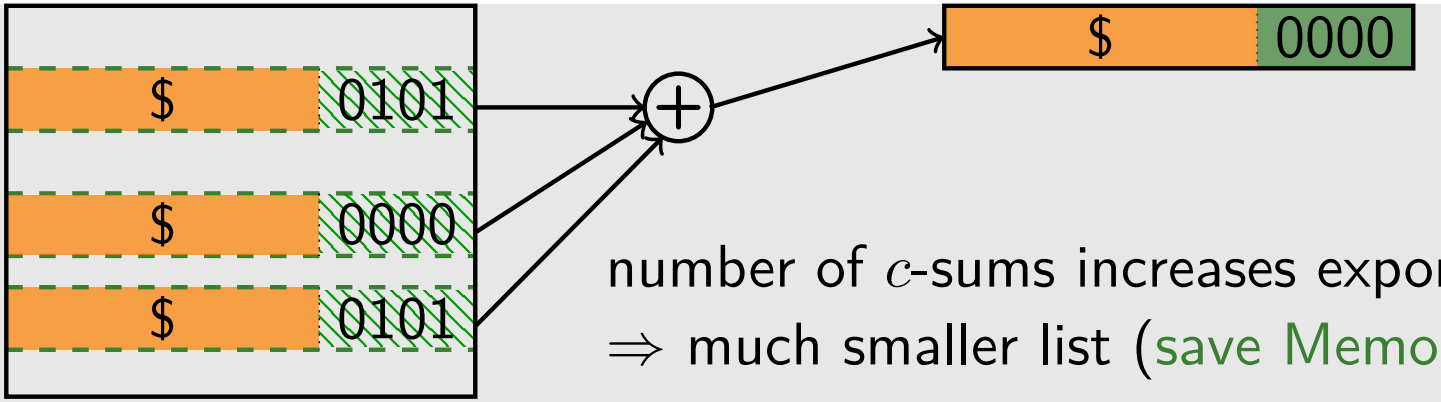

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$$N \geq 2^{b/(c-1)}$$

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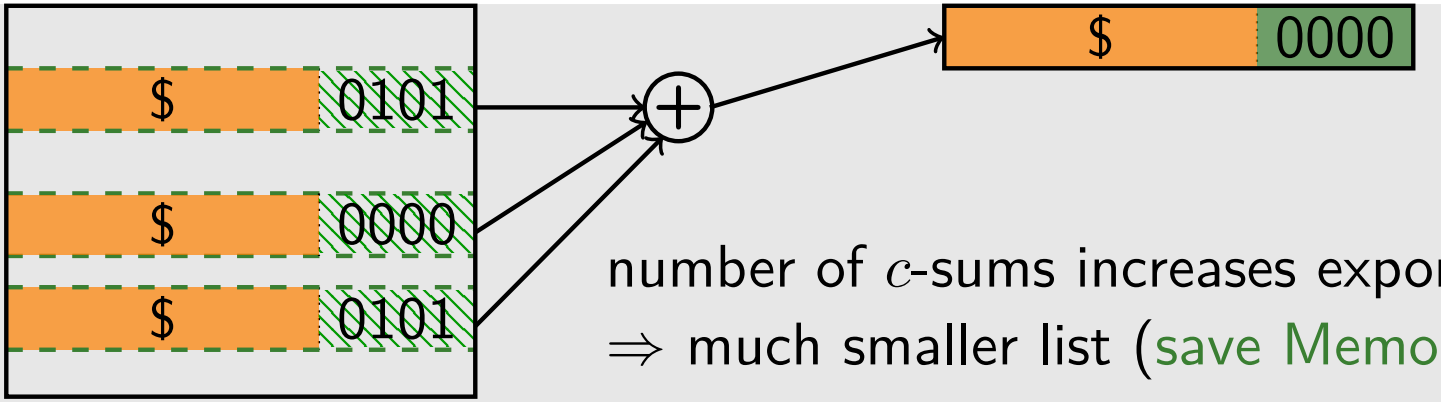

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$$N = 2^{b/(c-1)}$$

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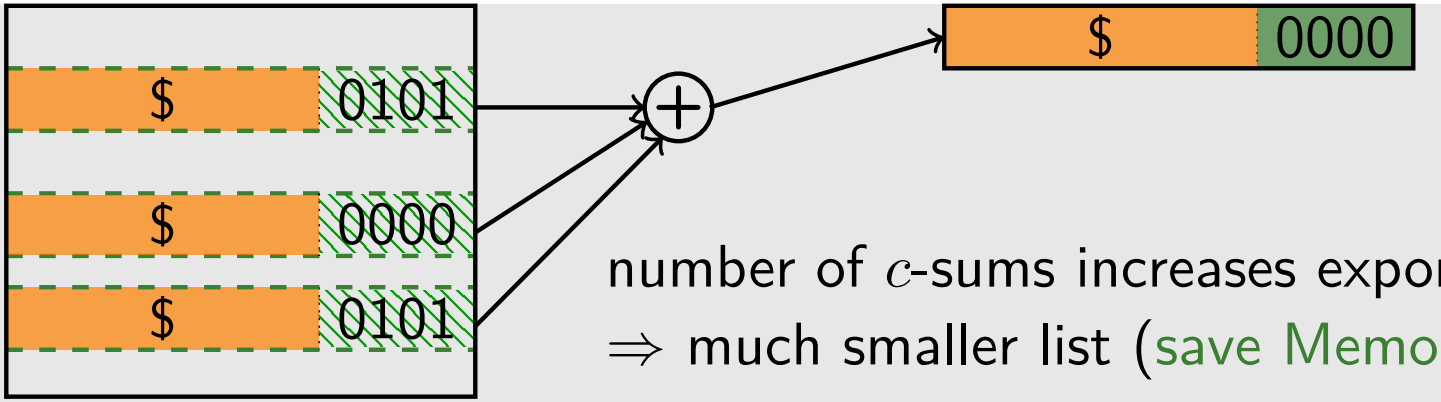

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Main Idea: solve c-SP repeatedly on stripes

$$N = 2^{b/(c-1)}$$

c-sum-Problem (c-SP)

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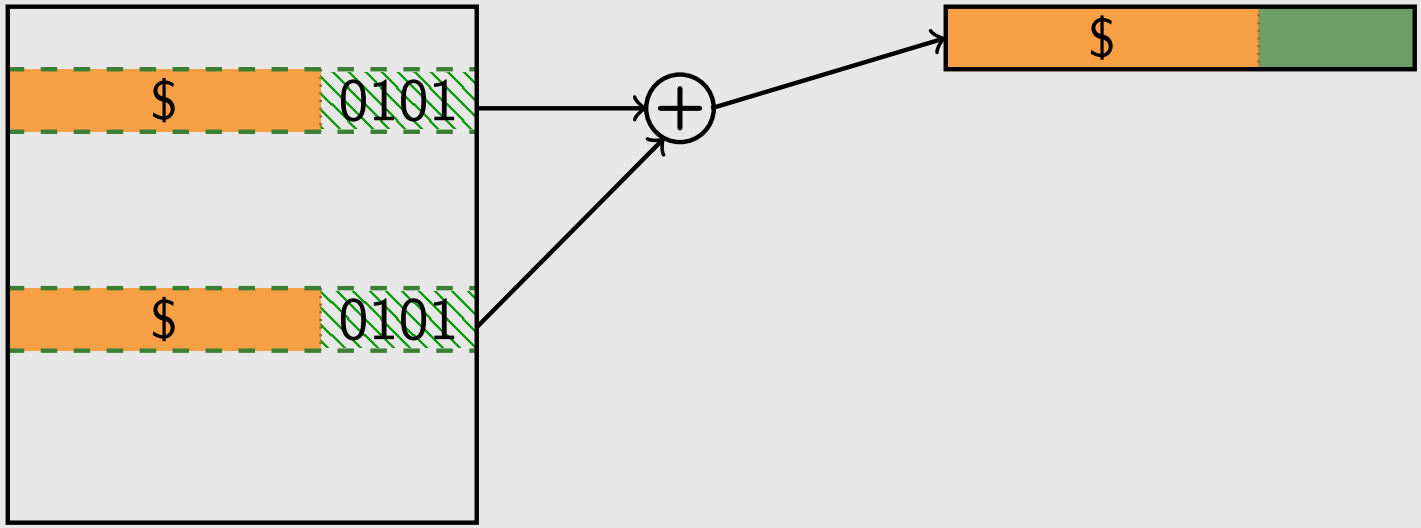
Not just a memory reduction technique



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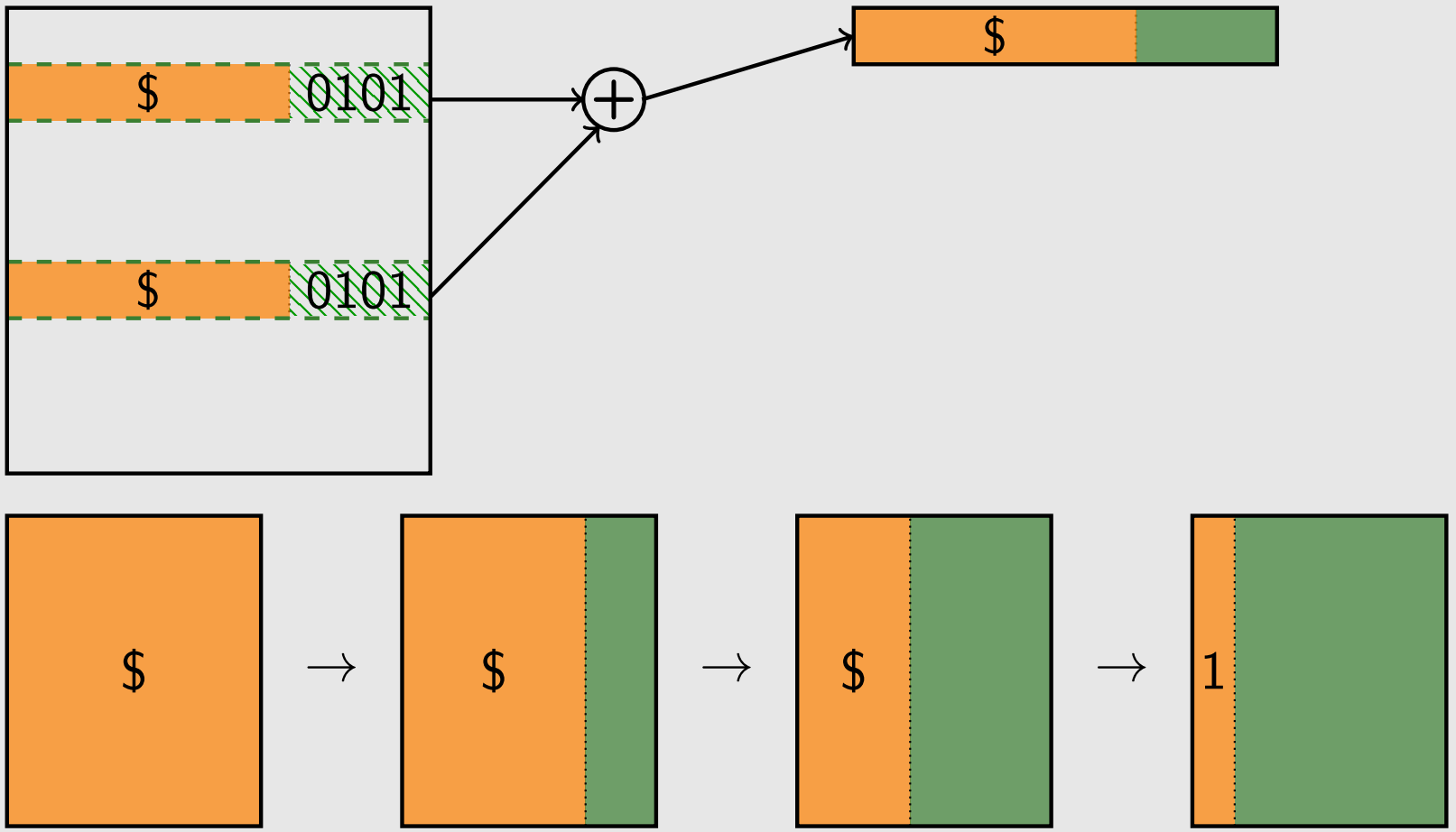

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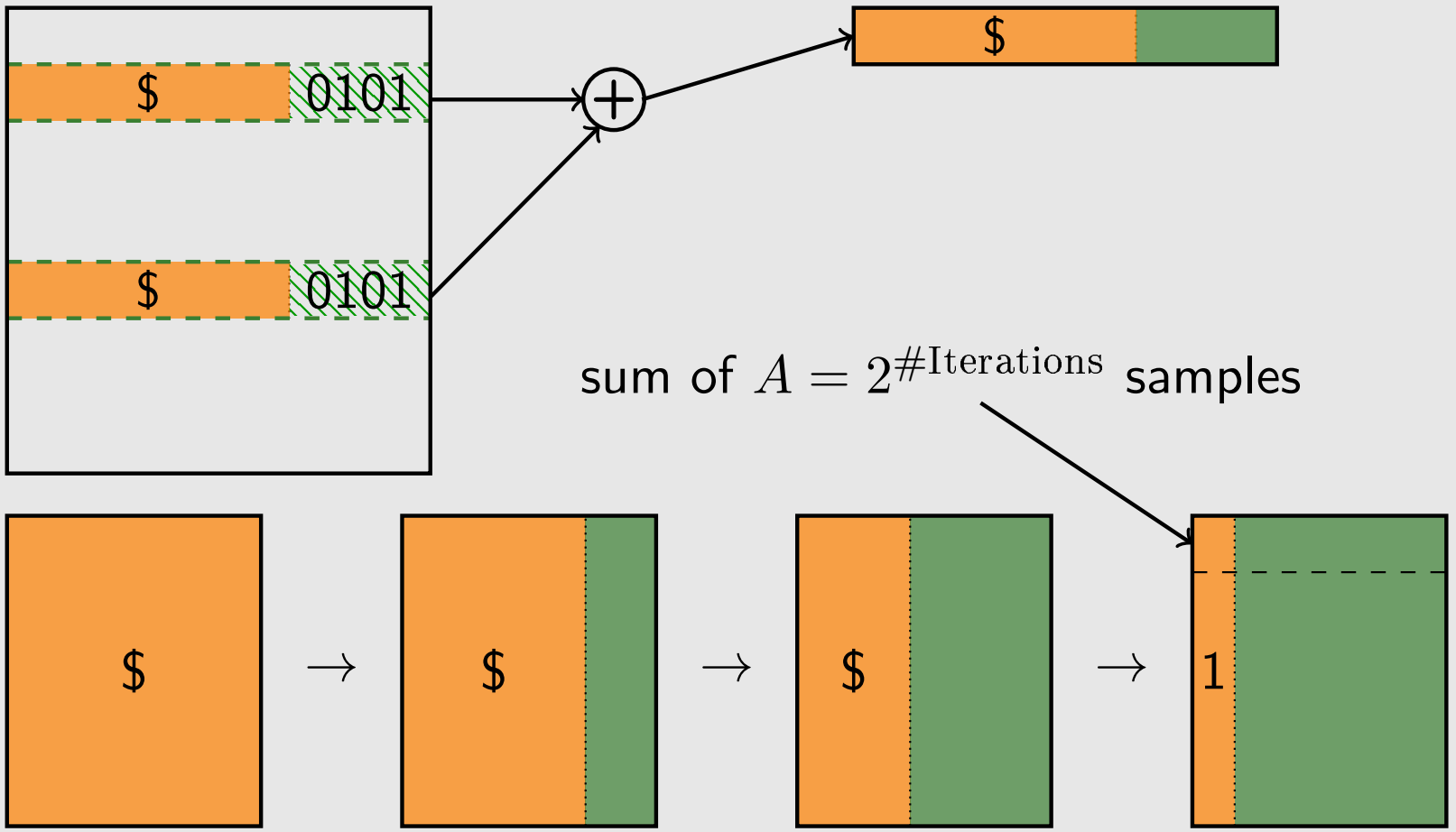

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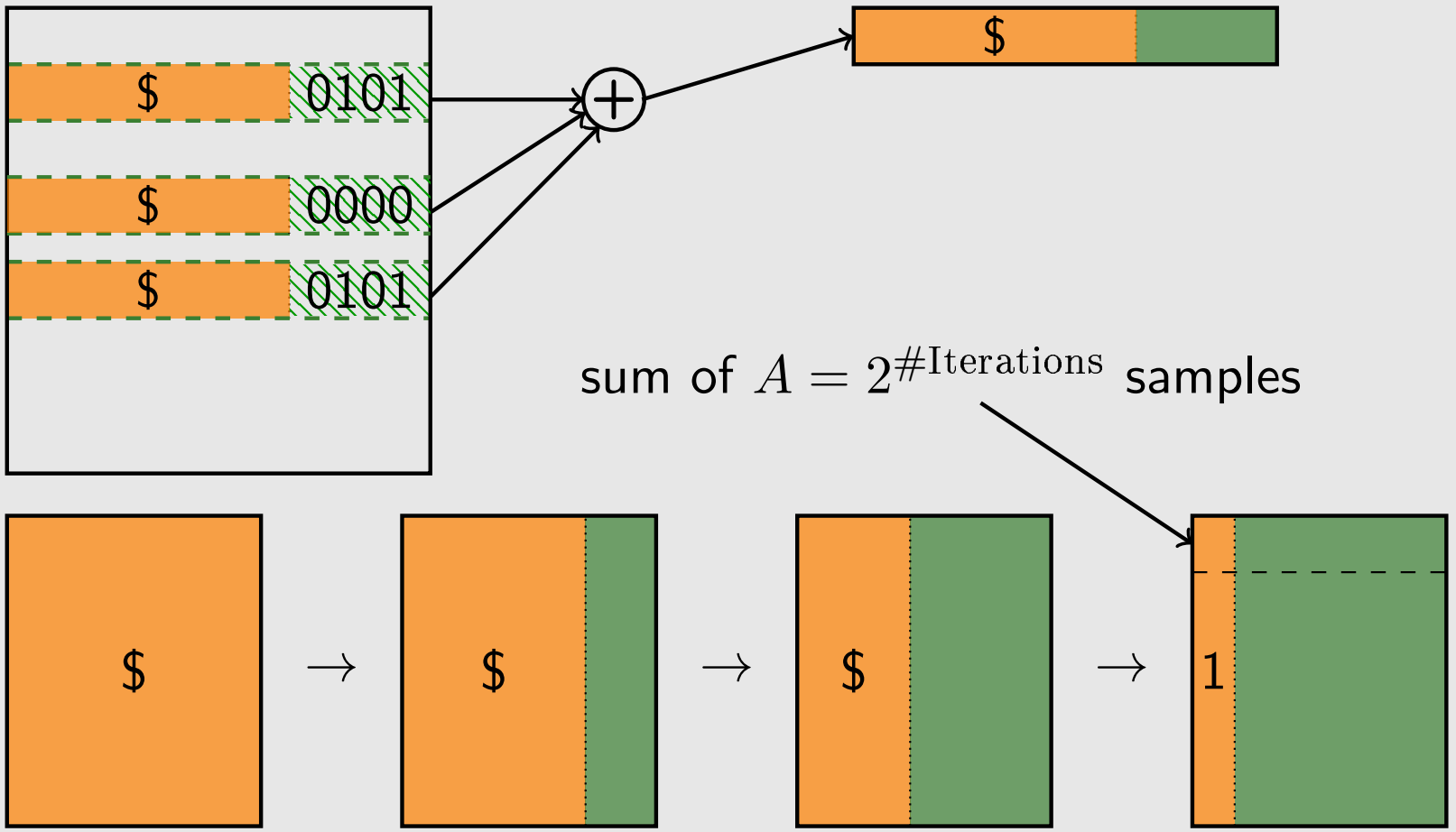

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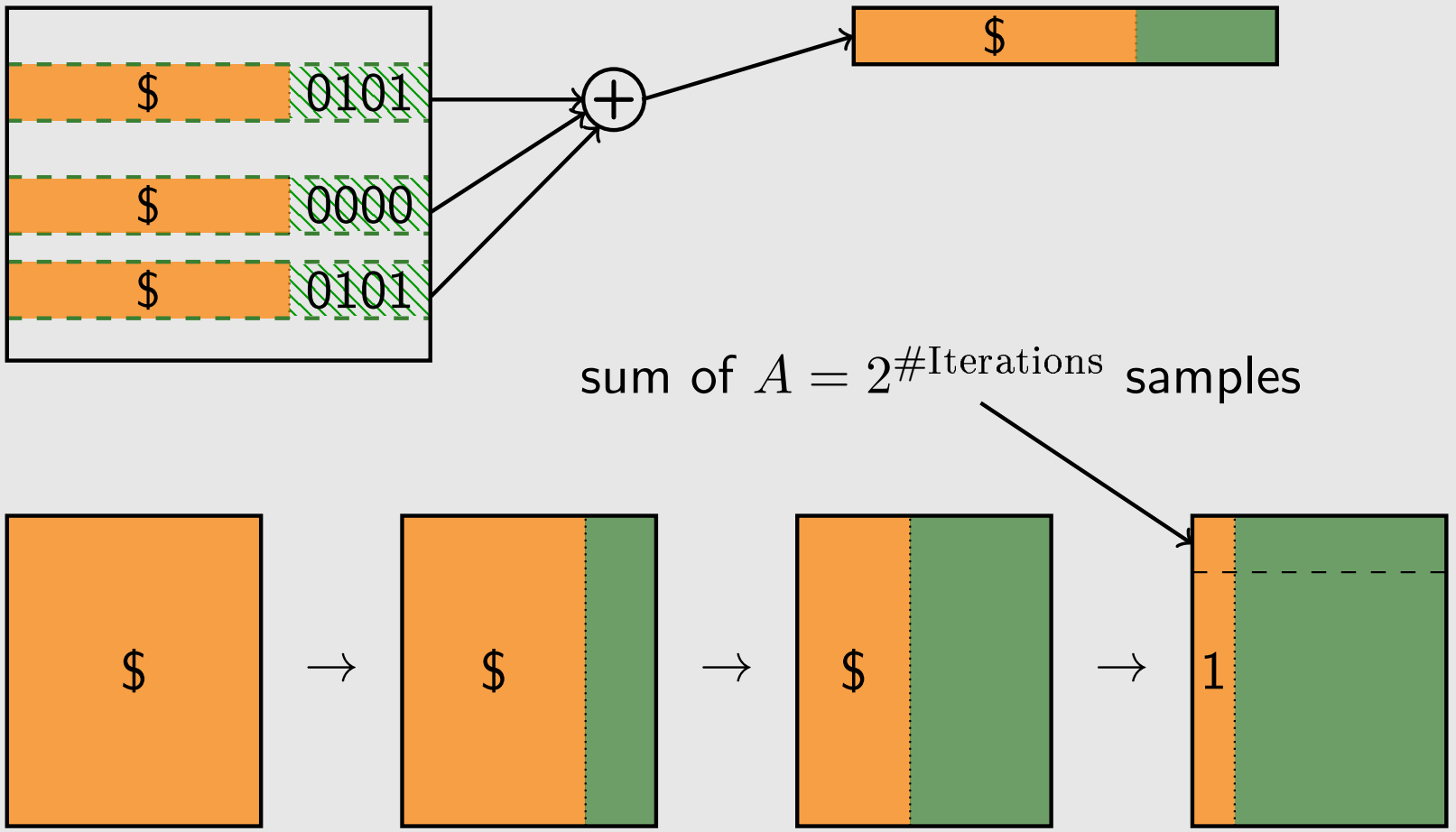

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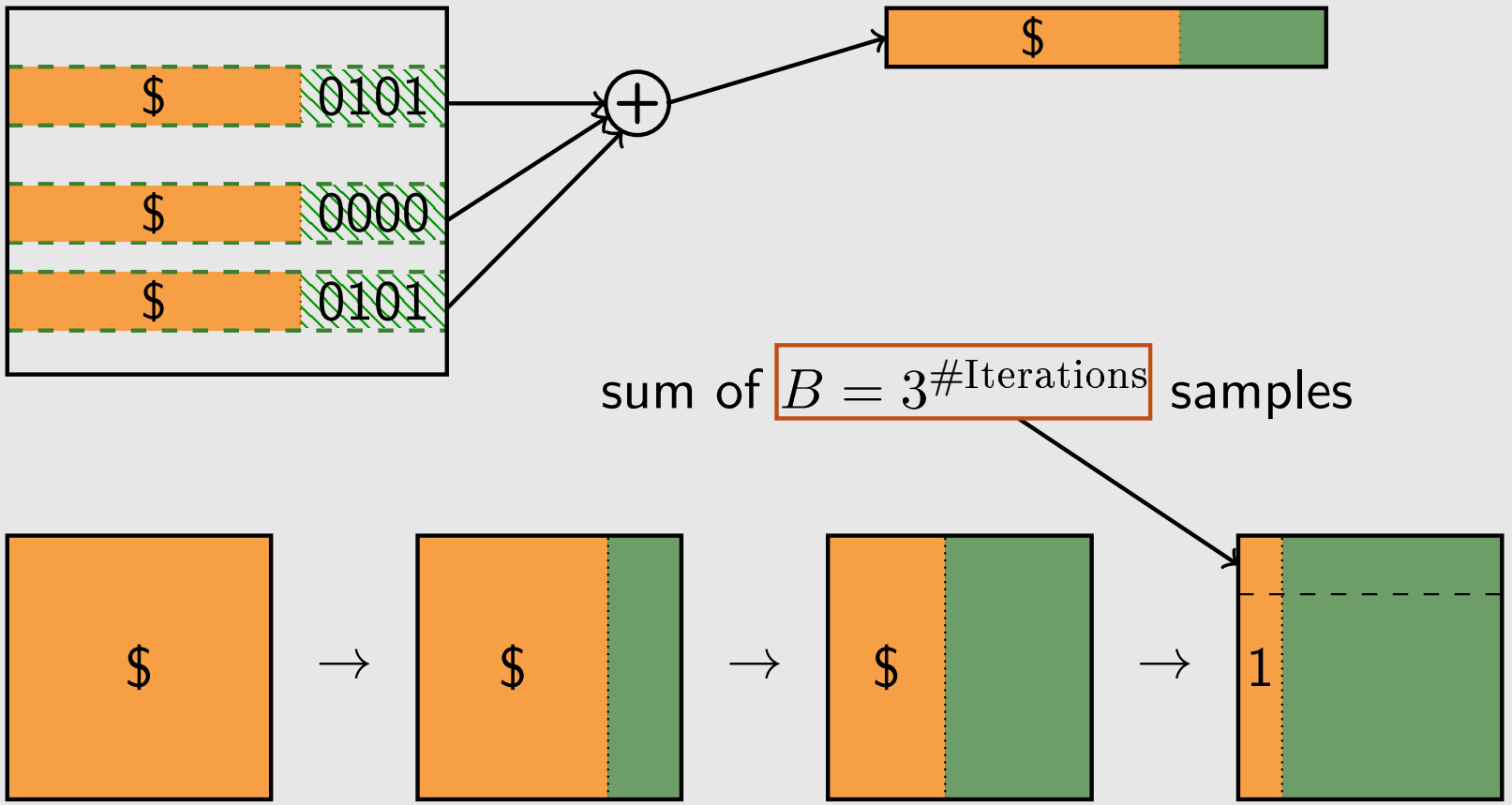

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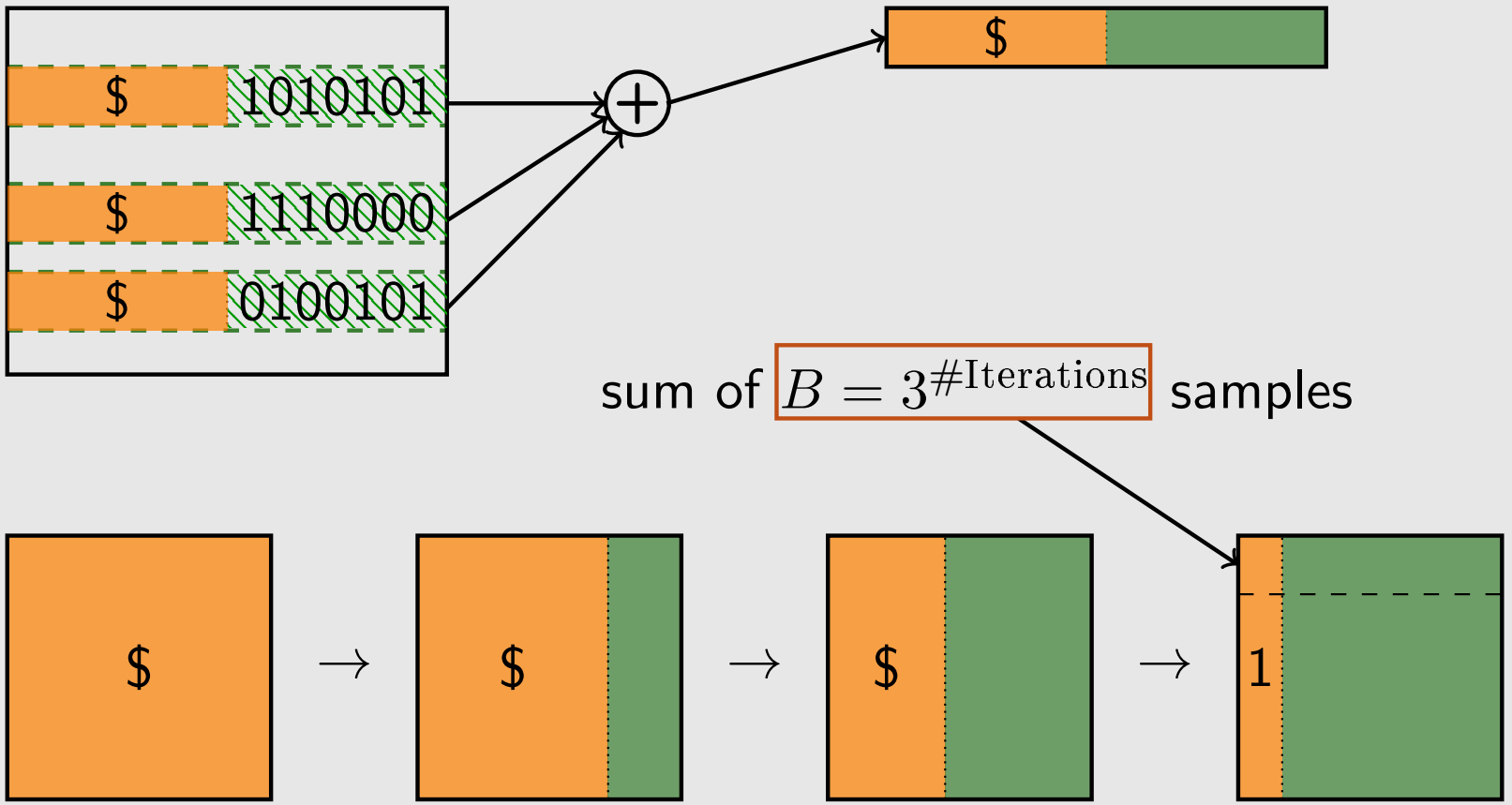

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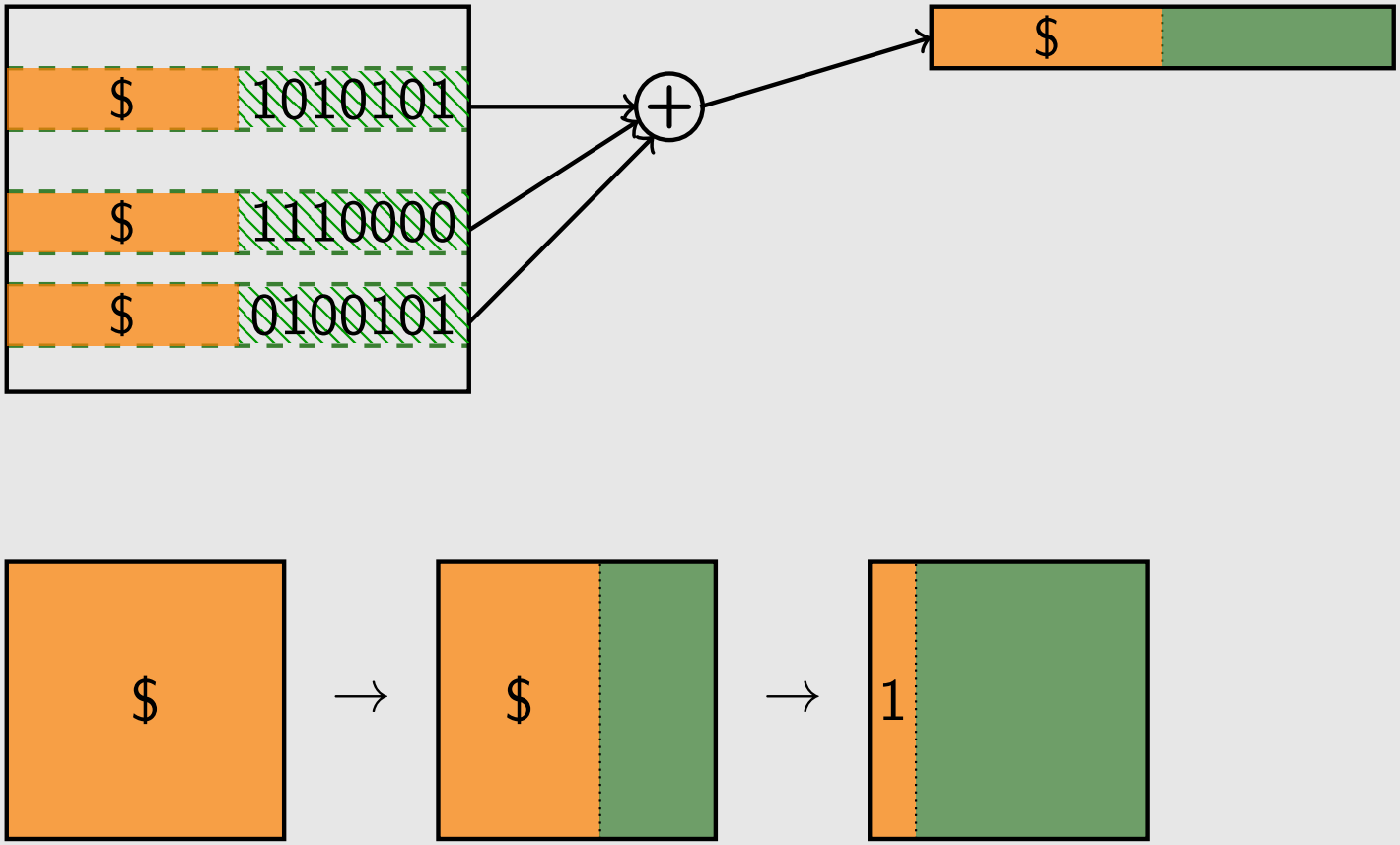

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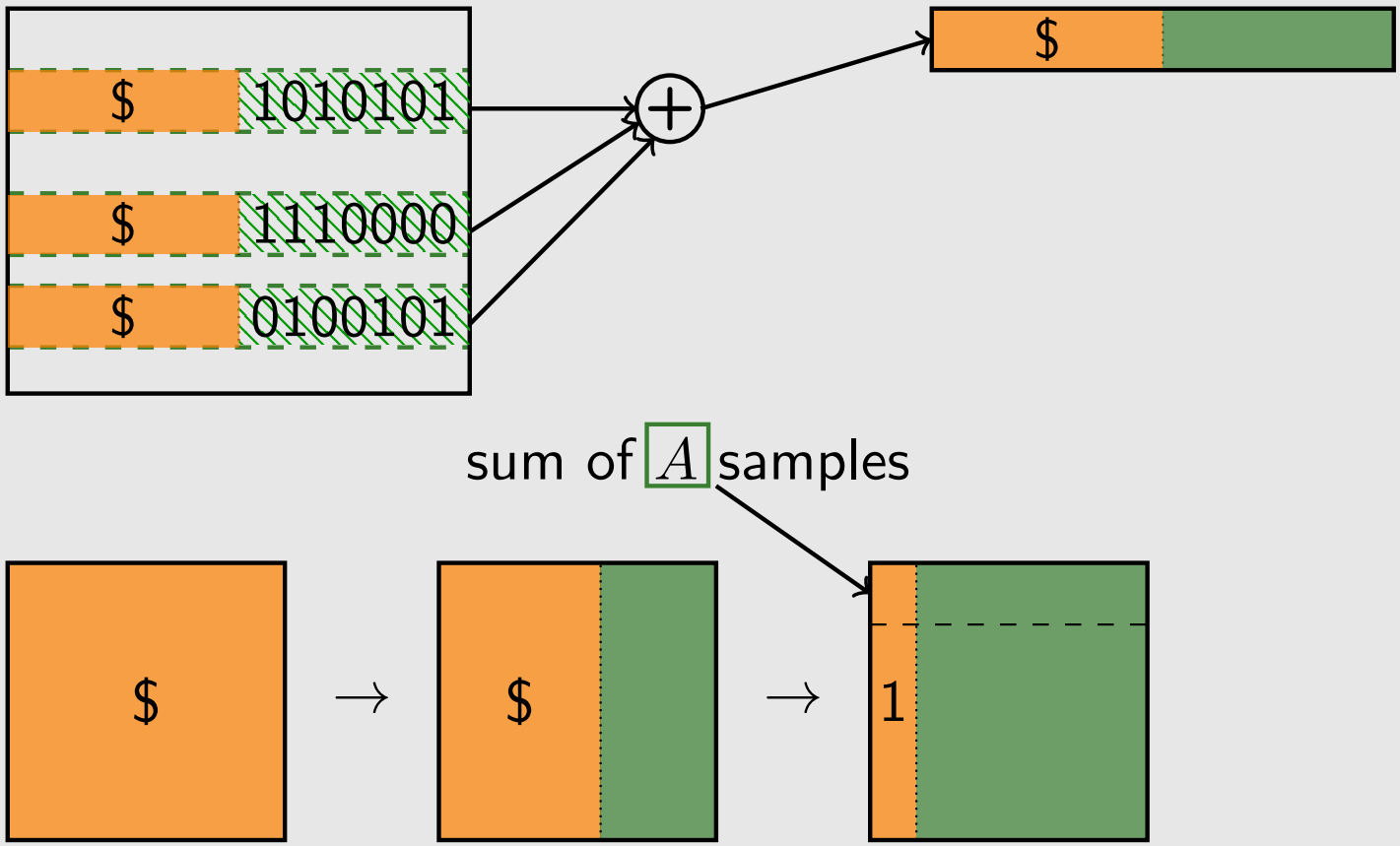

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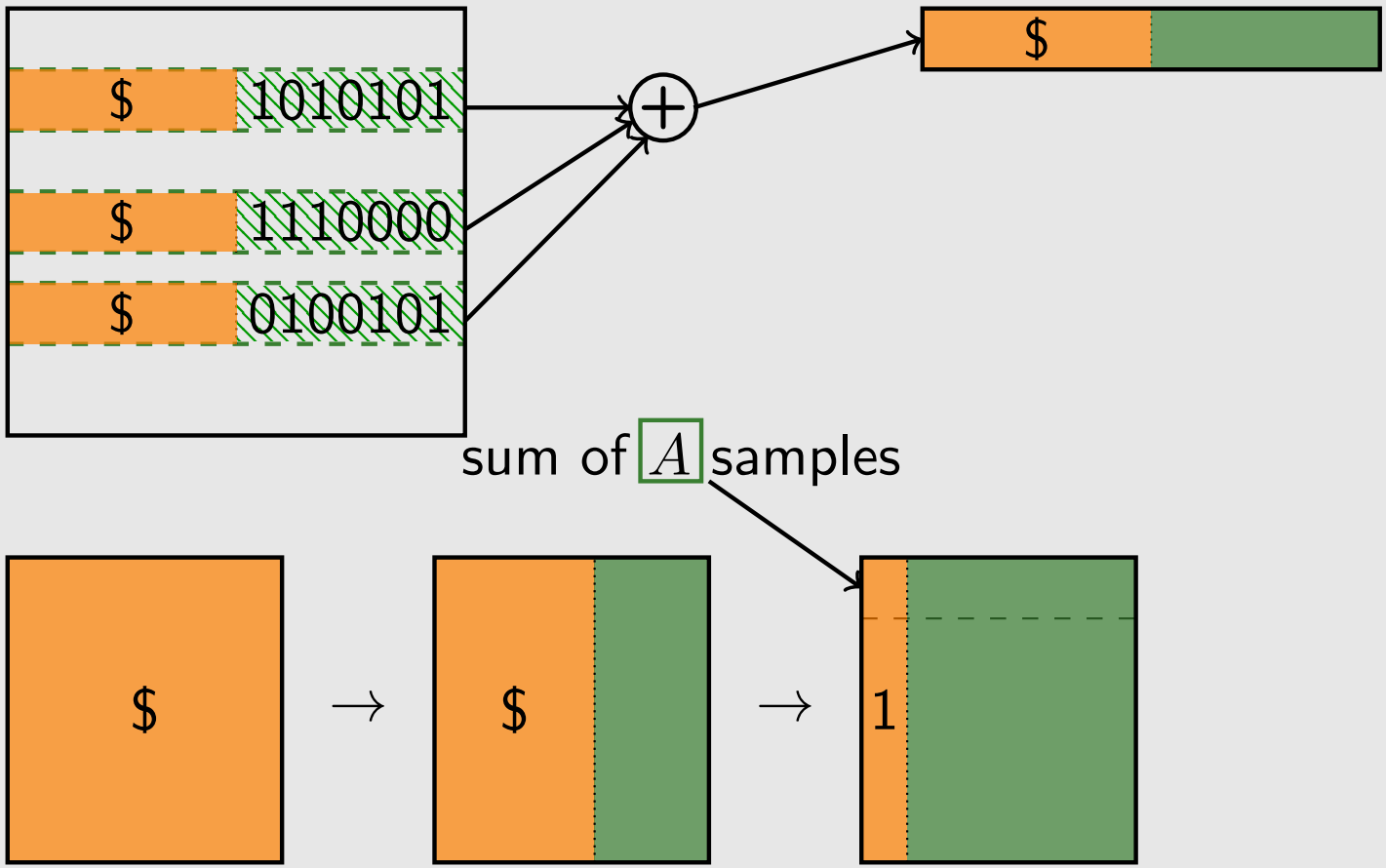

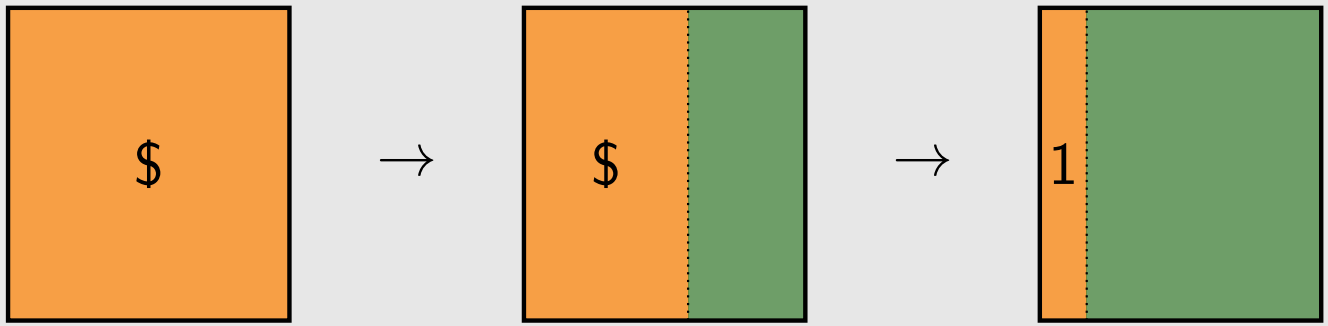
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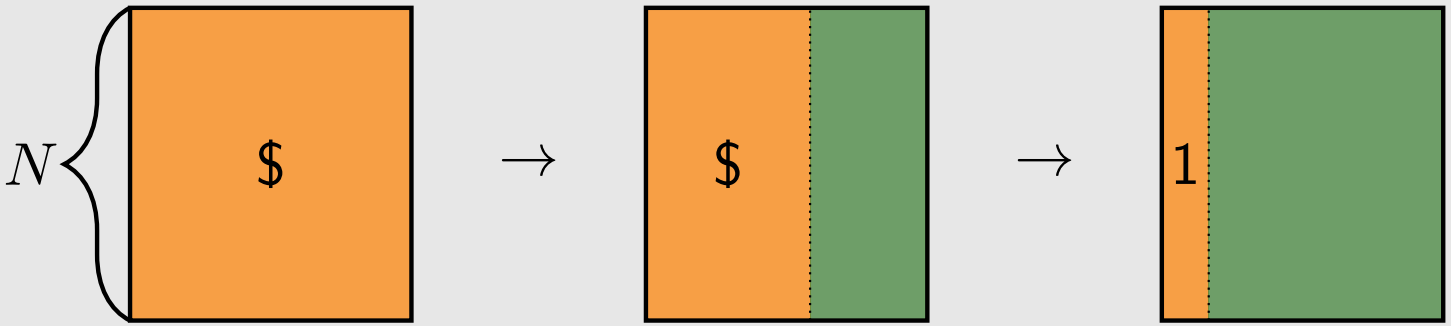
N
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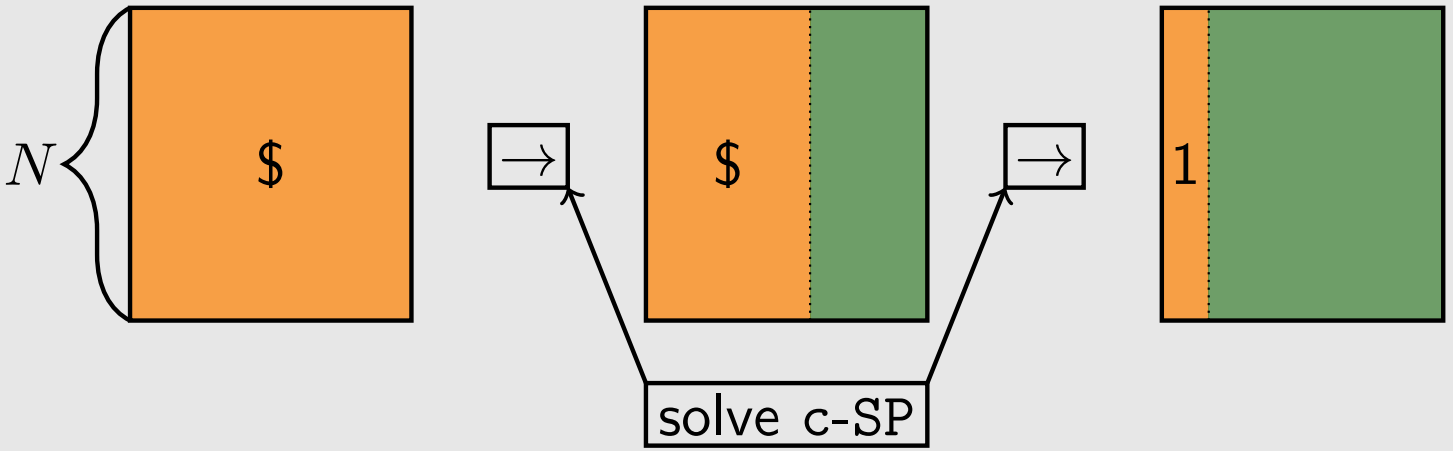
1


N
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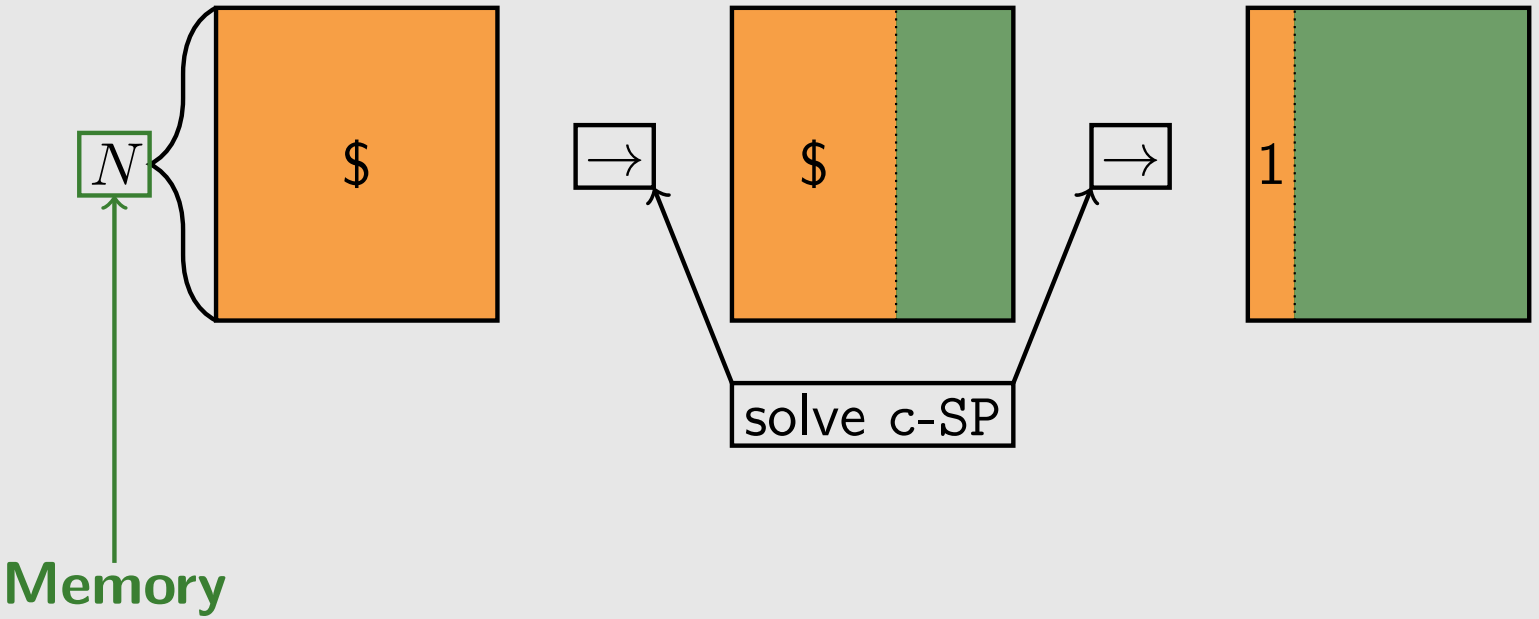
$
1
solve c-SP


N
$
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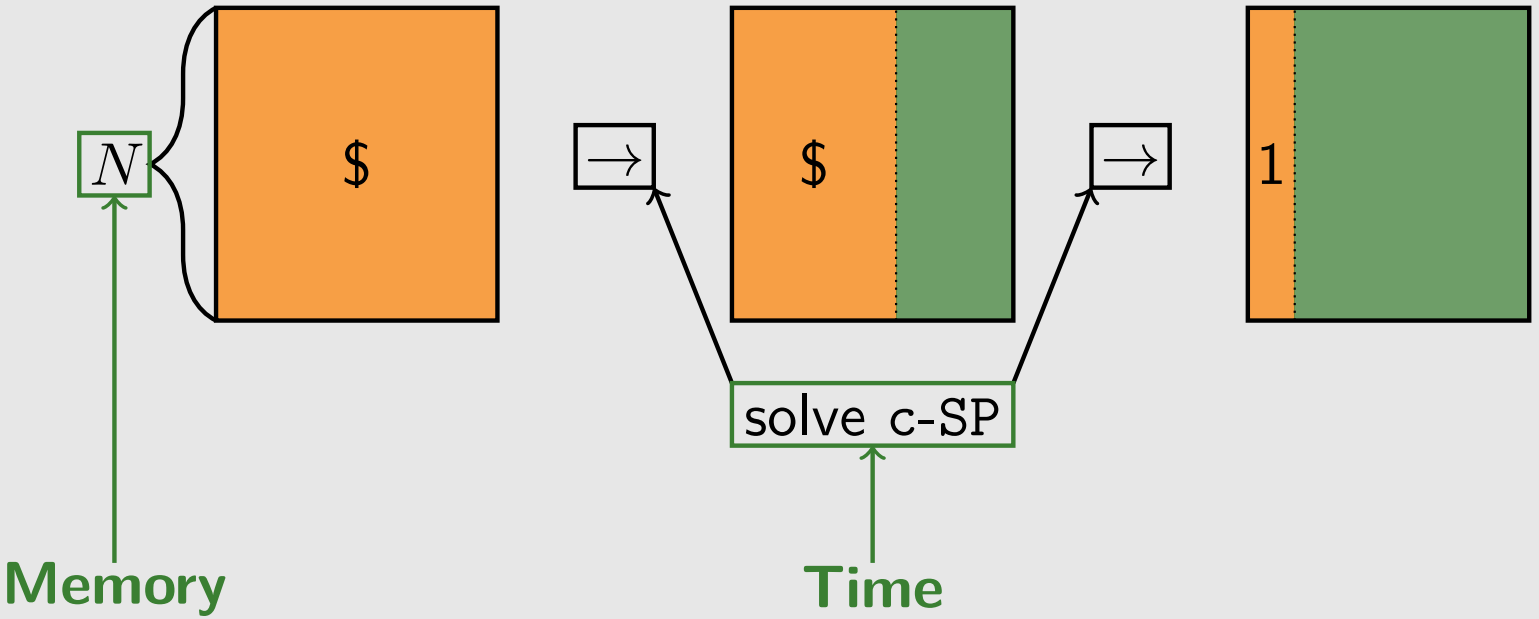
1
solve c-SP
Memory


N
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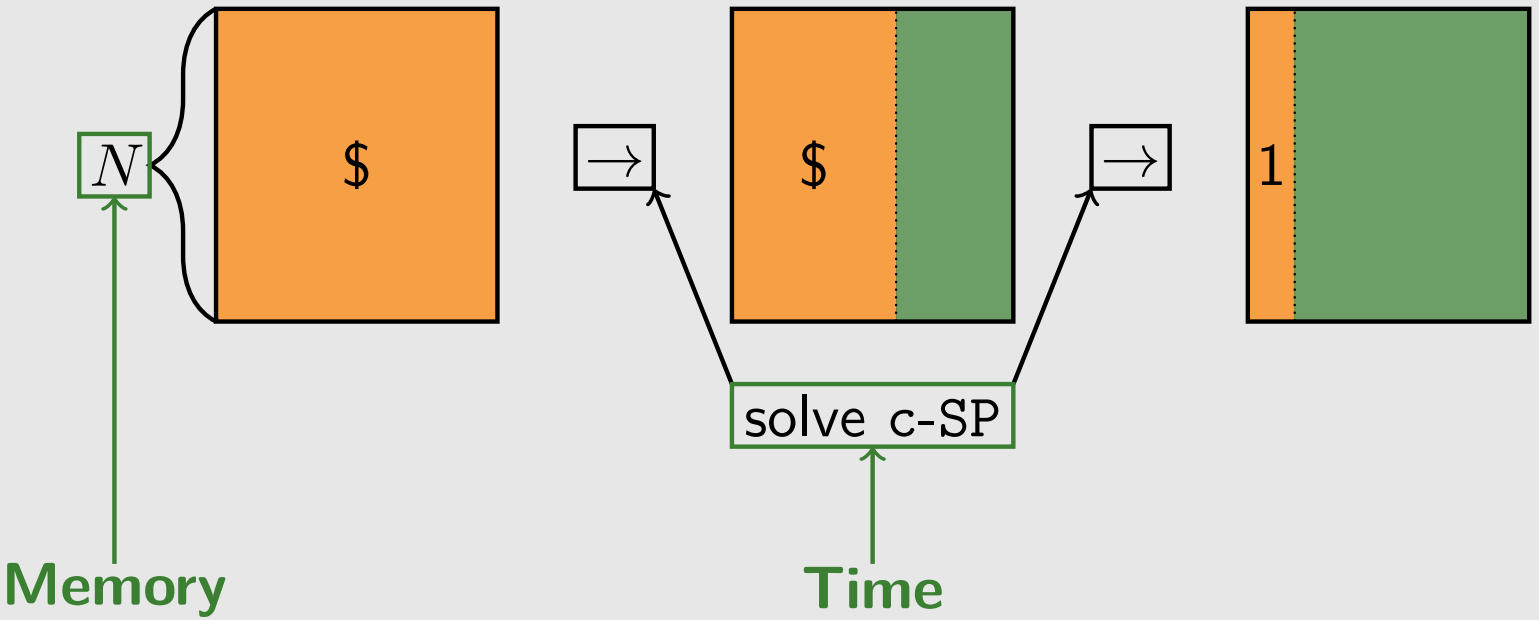
1
solve c-SP
Memory
Time



c-sum-BKW Theorem

LPN can be solved in time T and memory/samples M:

$$T = T_{\text{c-SP}} \text{ and } M = N$$

A naive Algorithm solving the c-sum-Problem

c-sum-naive Algorithm

Input : list L of size N
foreach $(c-1)$-*sum* x *of* L **do**
  if $x \in L$ **then**
    save c-sum

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c-sum-naive Theorem

c-`sum-naive` solves the c-sum-Problem in time T and Memory M:

$$T = N^{c-1} \text{ and } M = N$$

First TMTO for BKW

c-sum-naive-BKW Theorem

`c-sum-naive-BKW` solves LPN in time T and memory/samples M:

$$\log T = \log c \cdot \frac{k}{\log k} \quad \text{and} \quad \log M = \frac{\log c}{c-1} \cdot \frac{k}{\log k}$$

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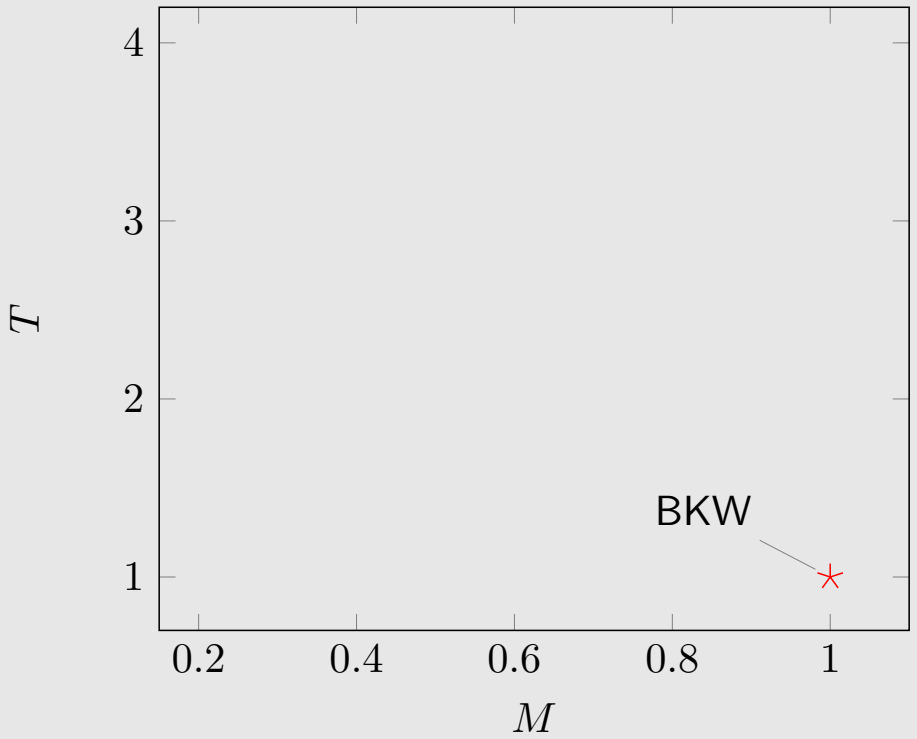

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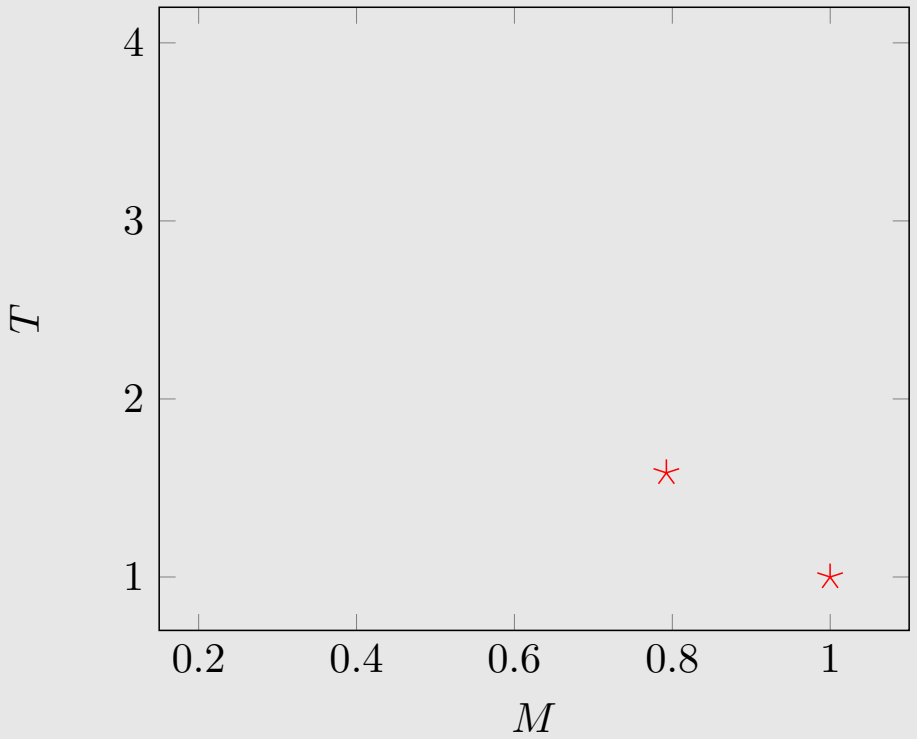

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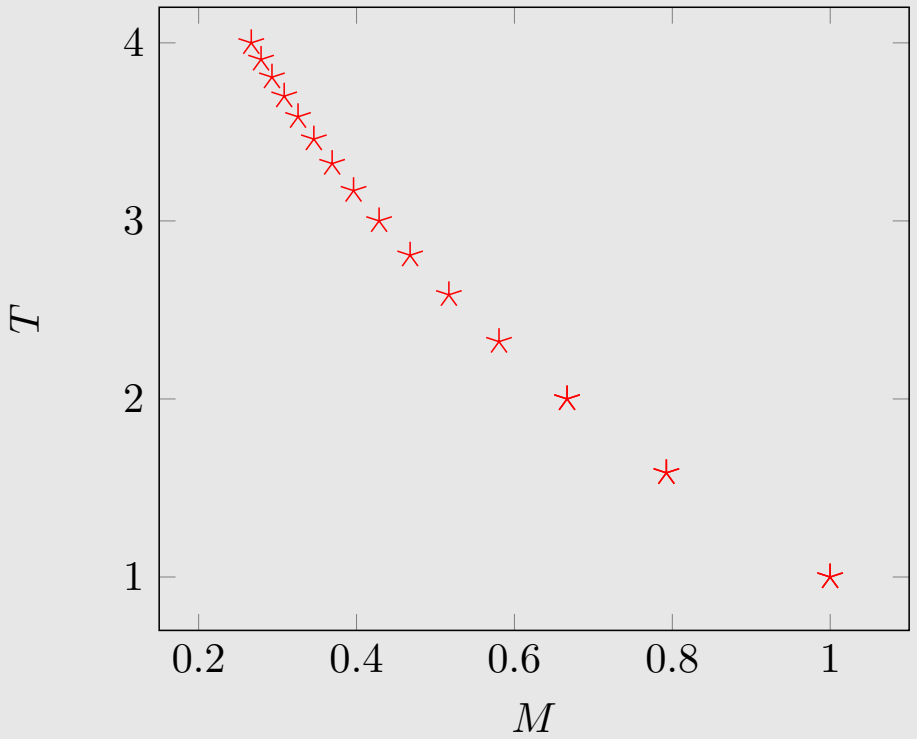
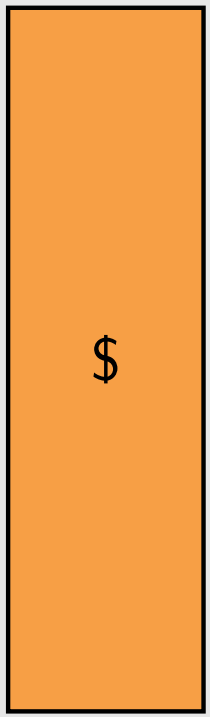


The Idea of Schroeppel-Shamir



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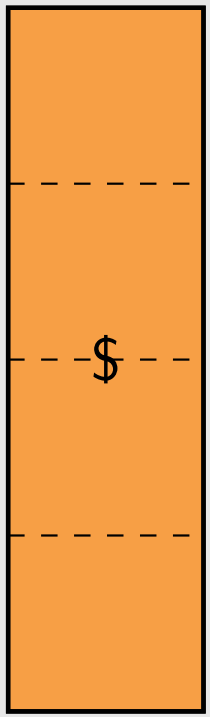

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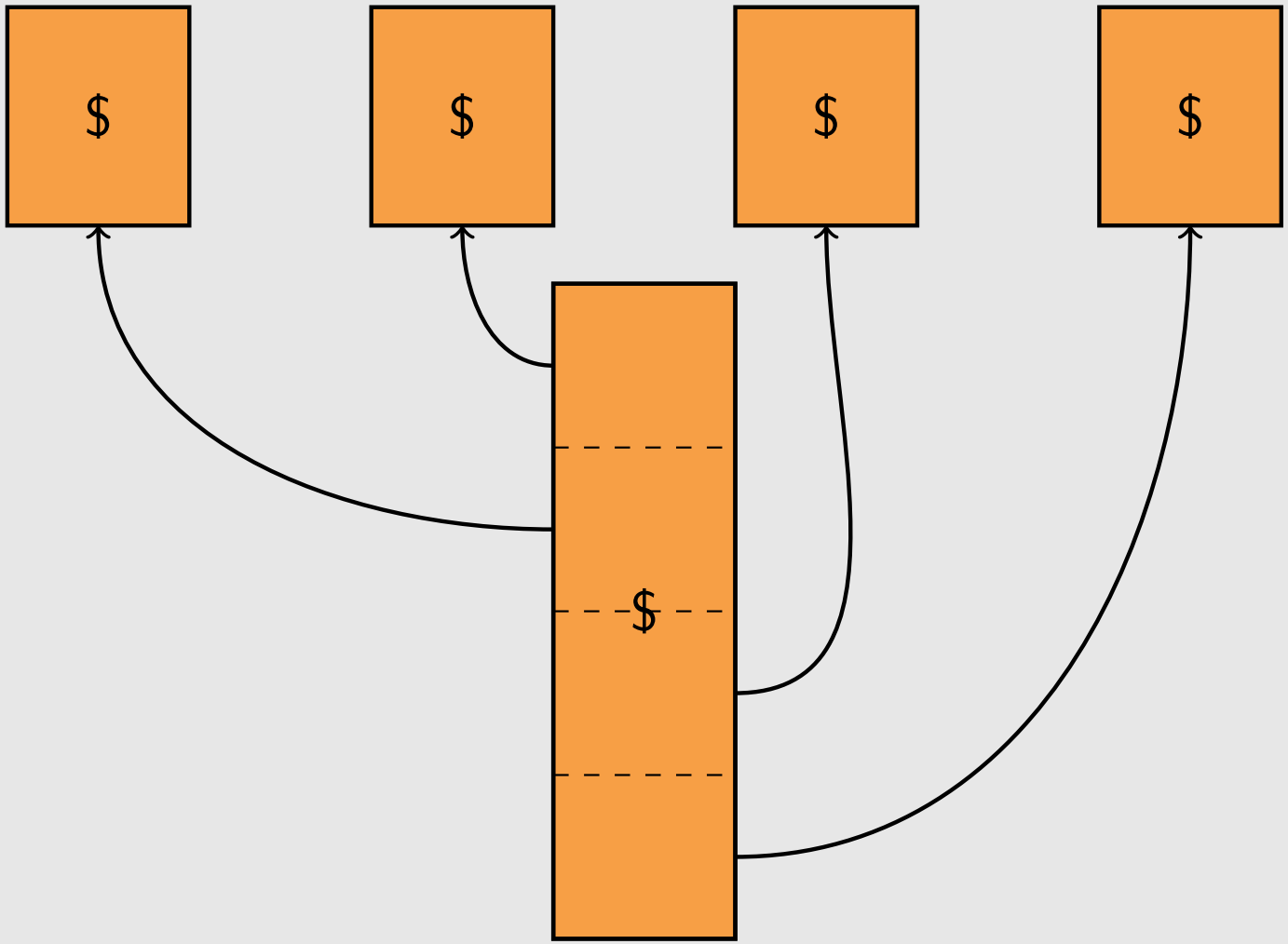

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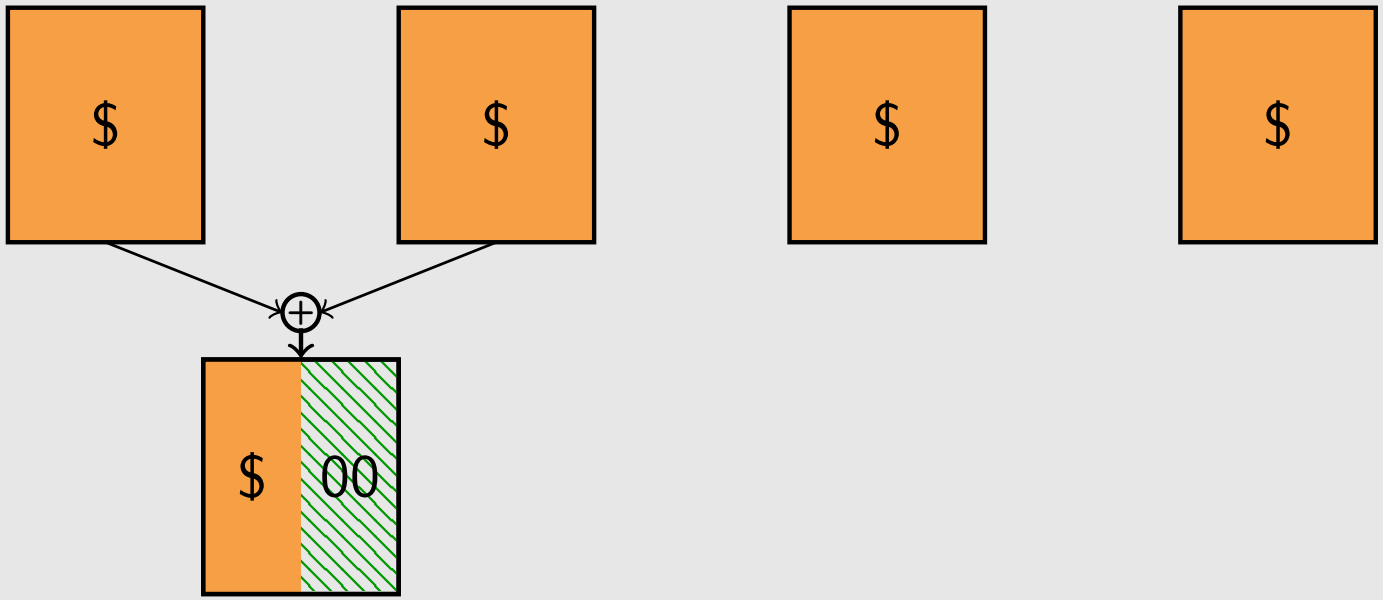

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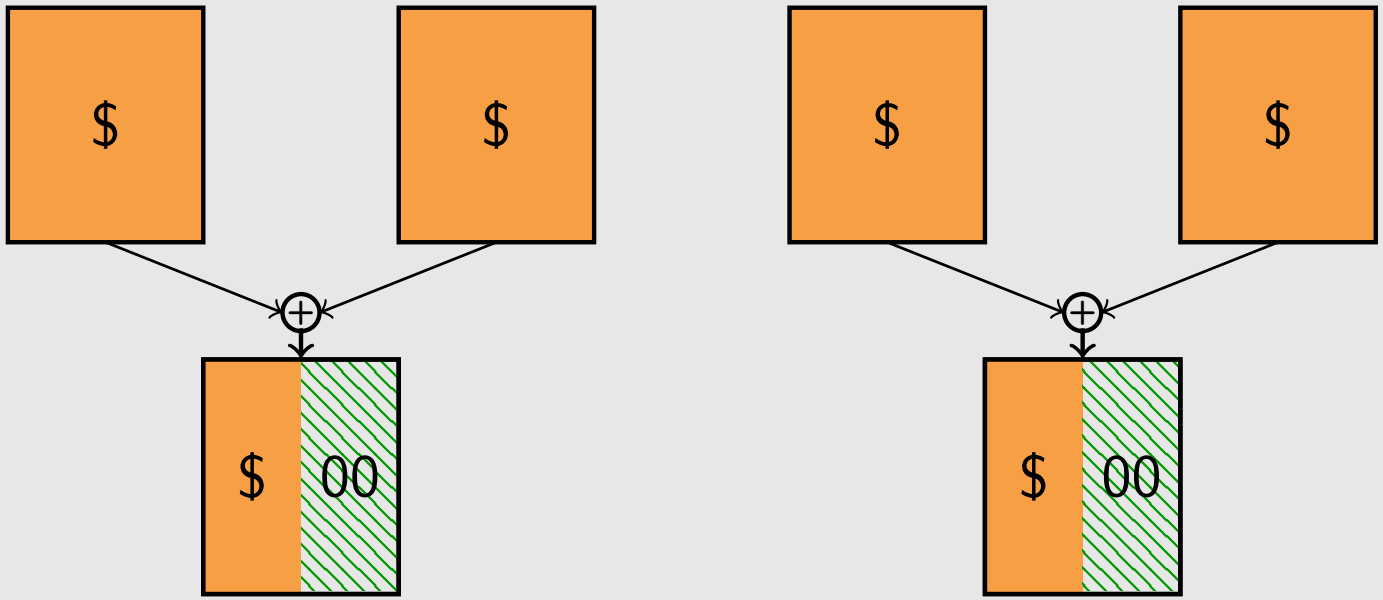

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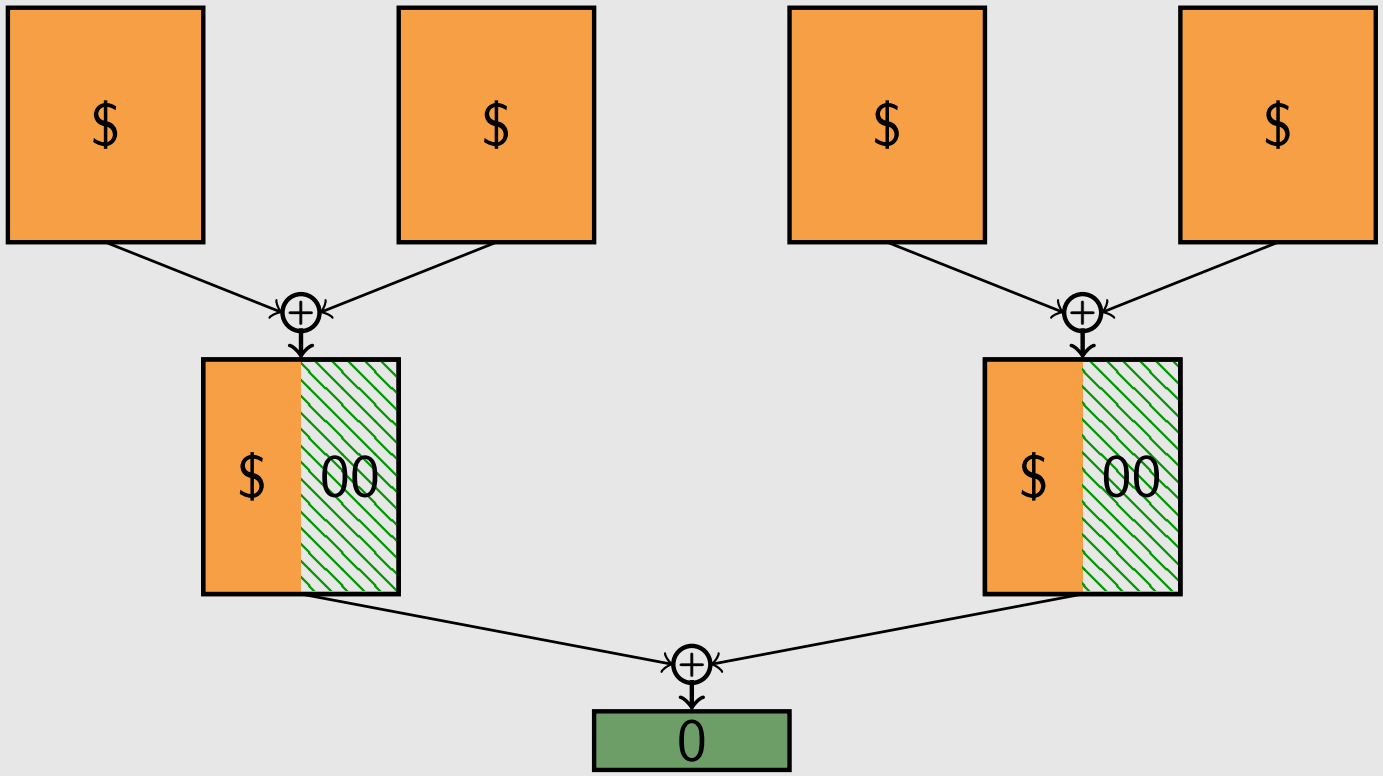

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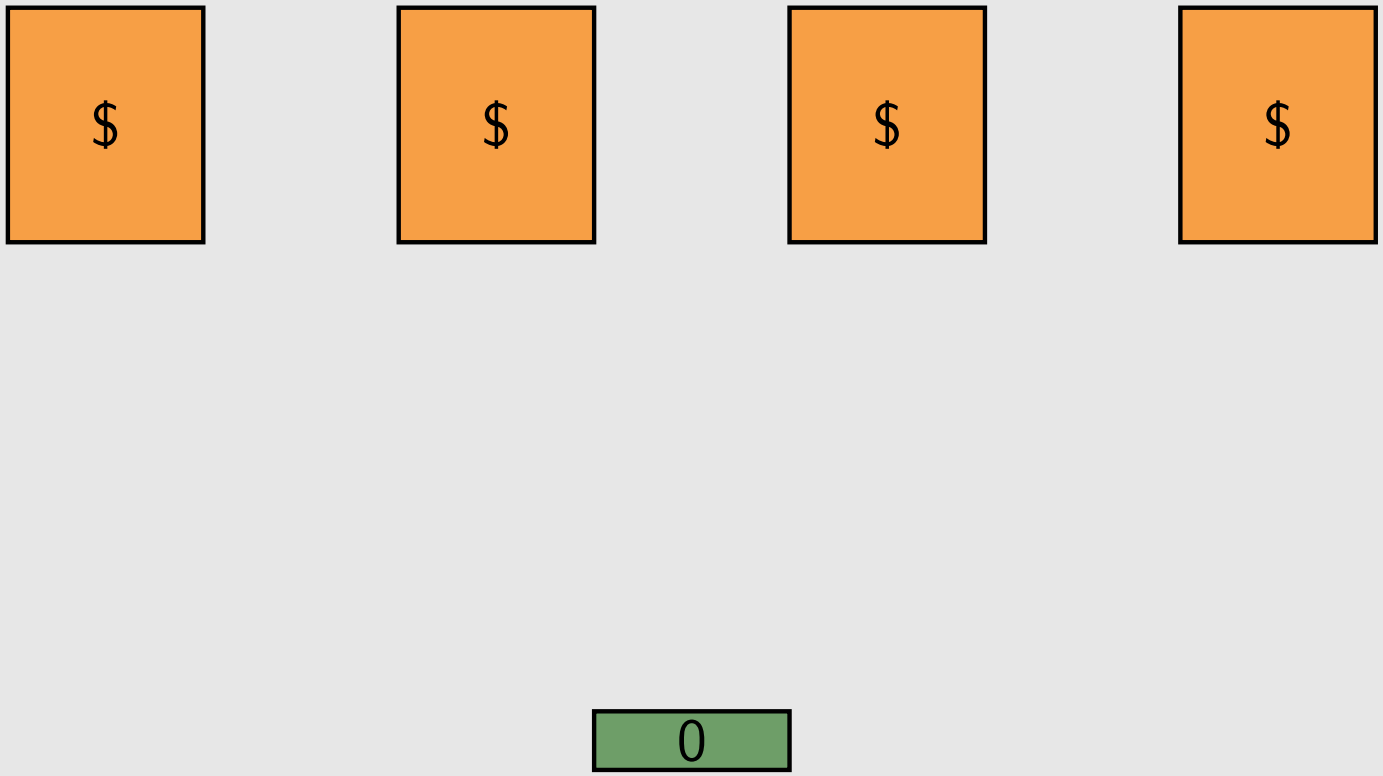

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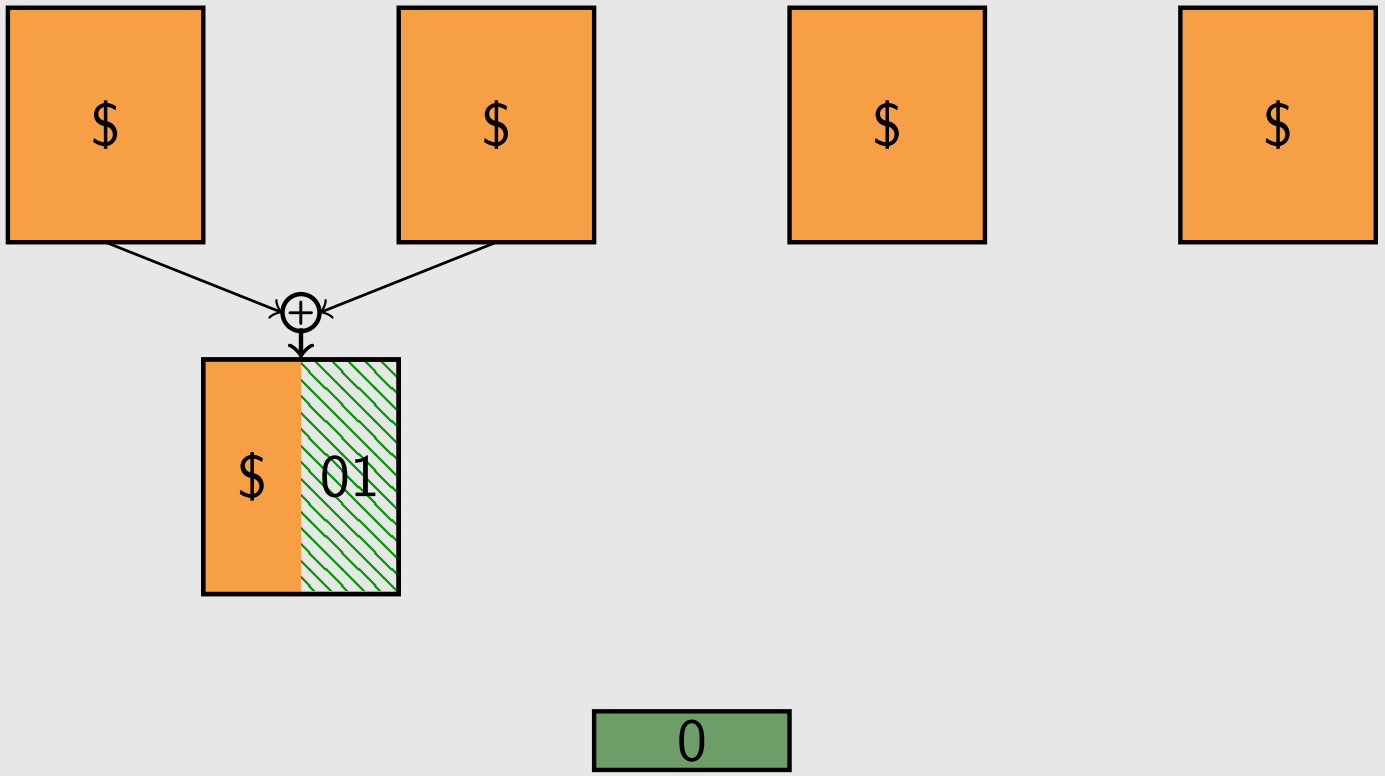

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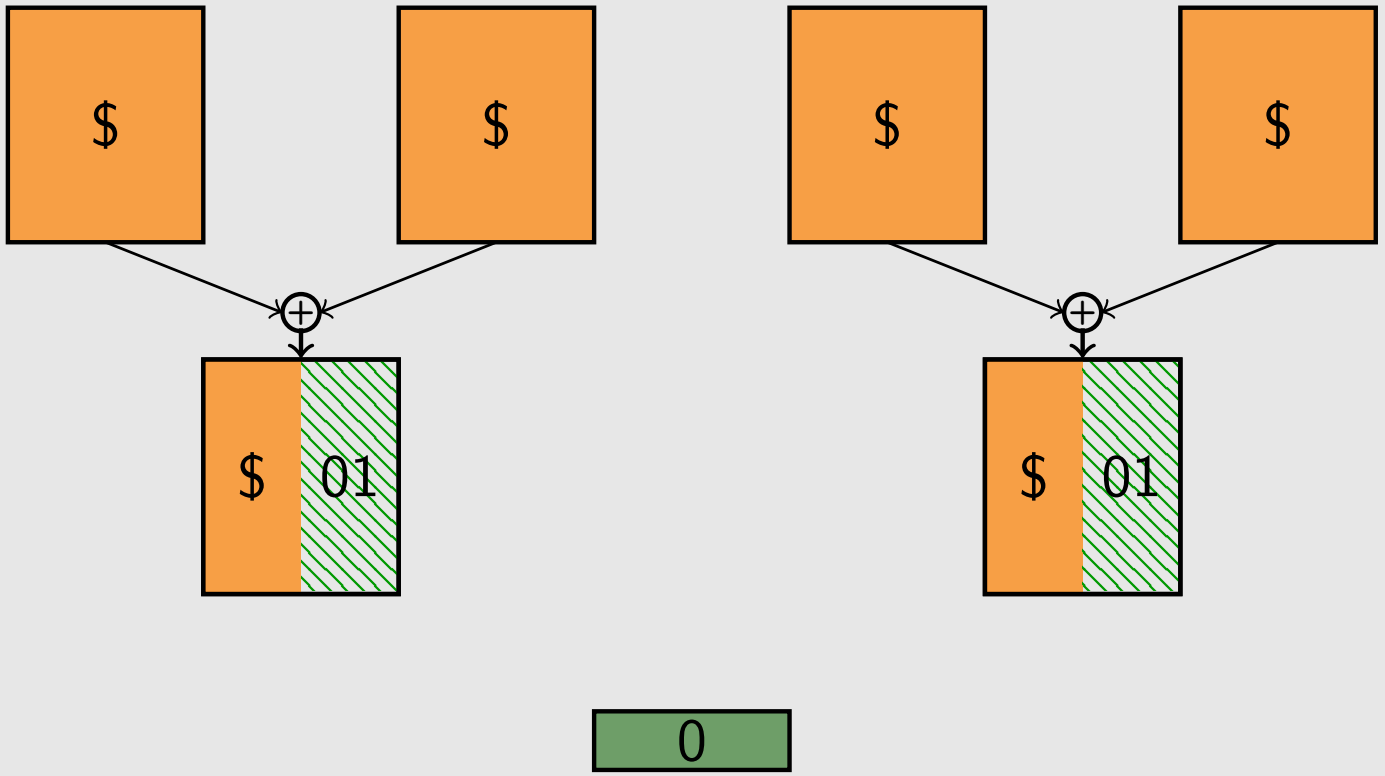

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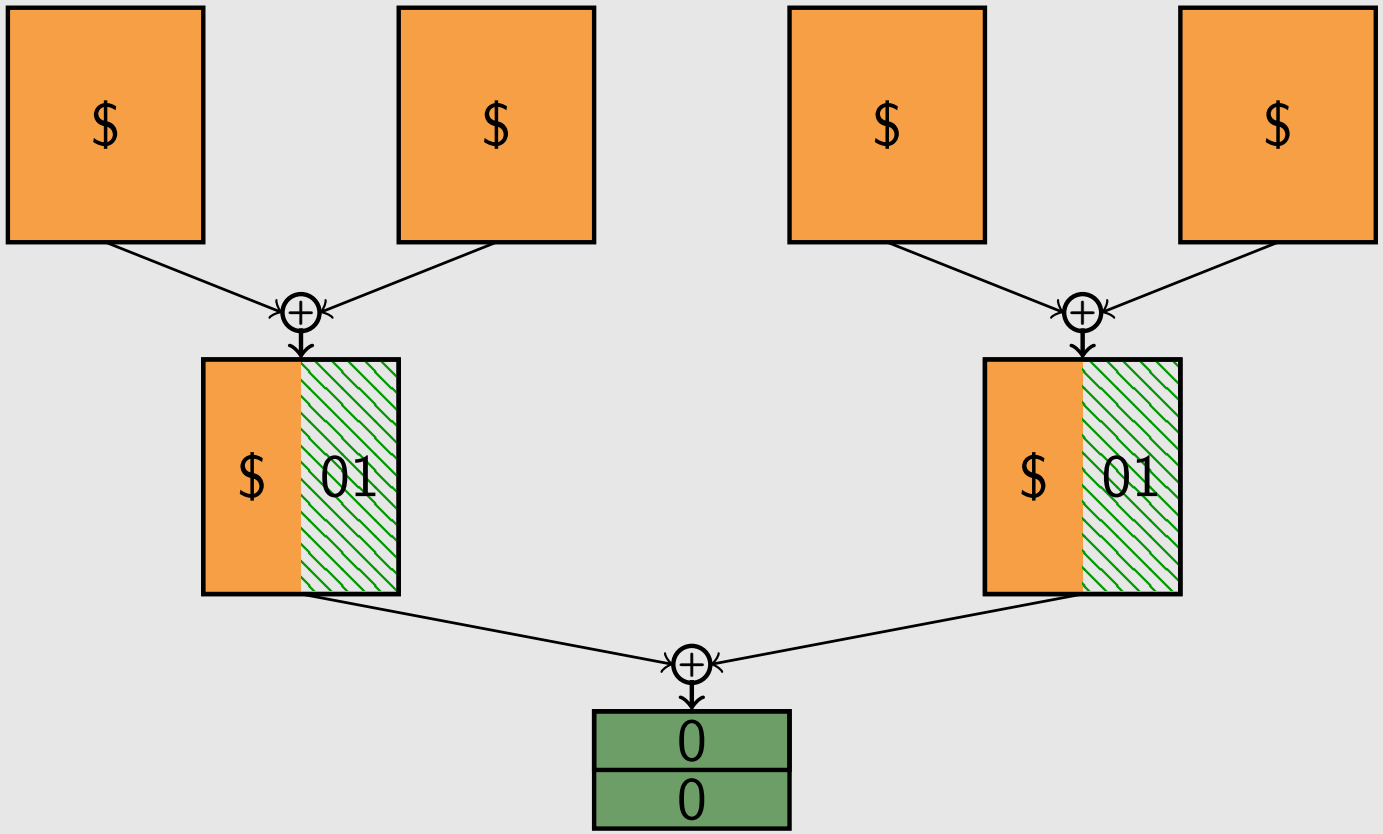

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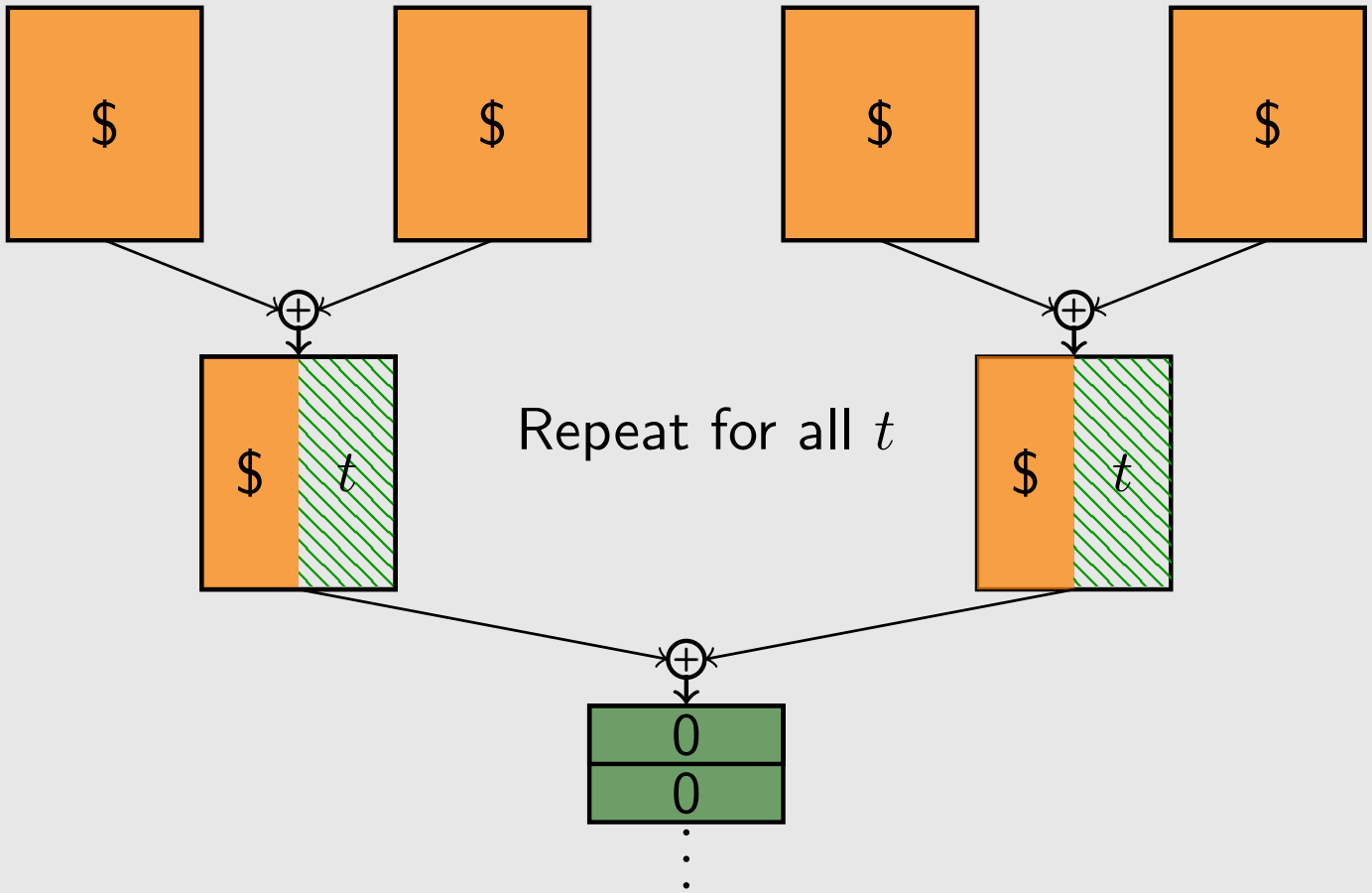

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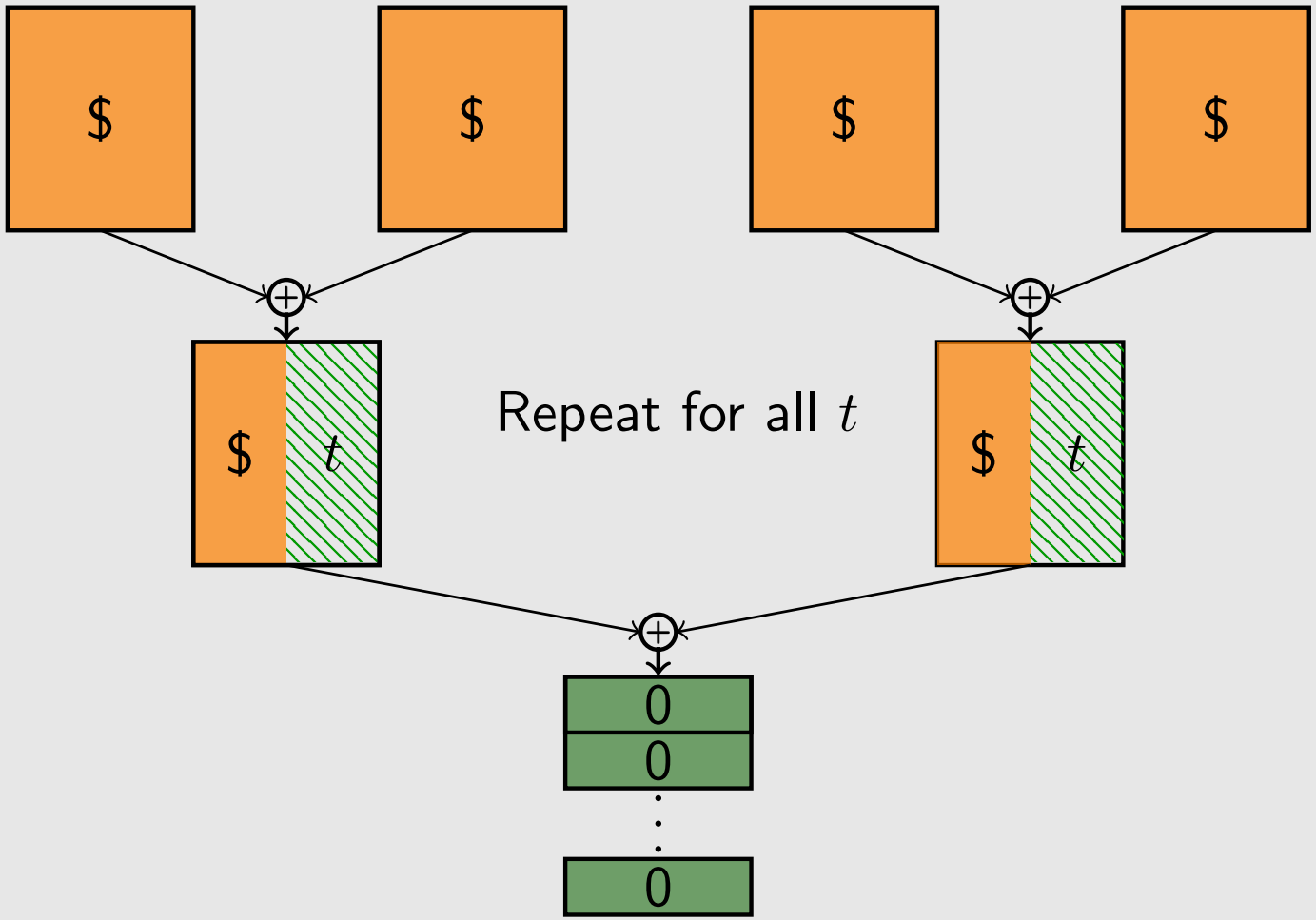

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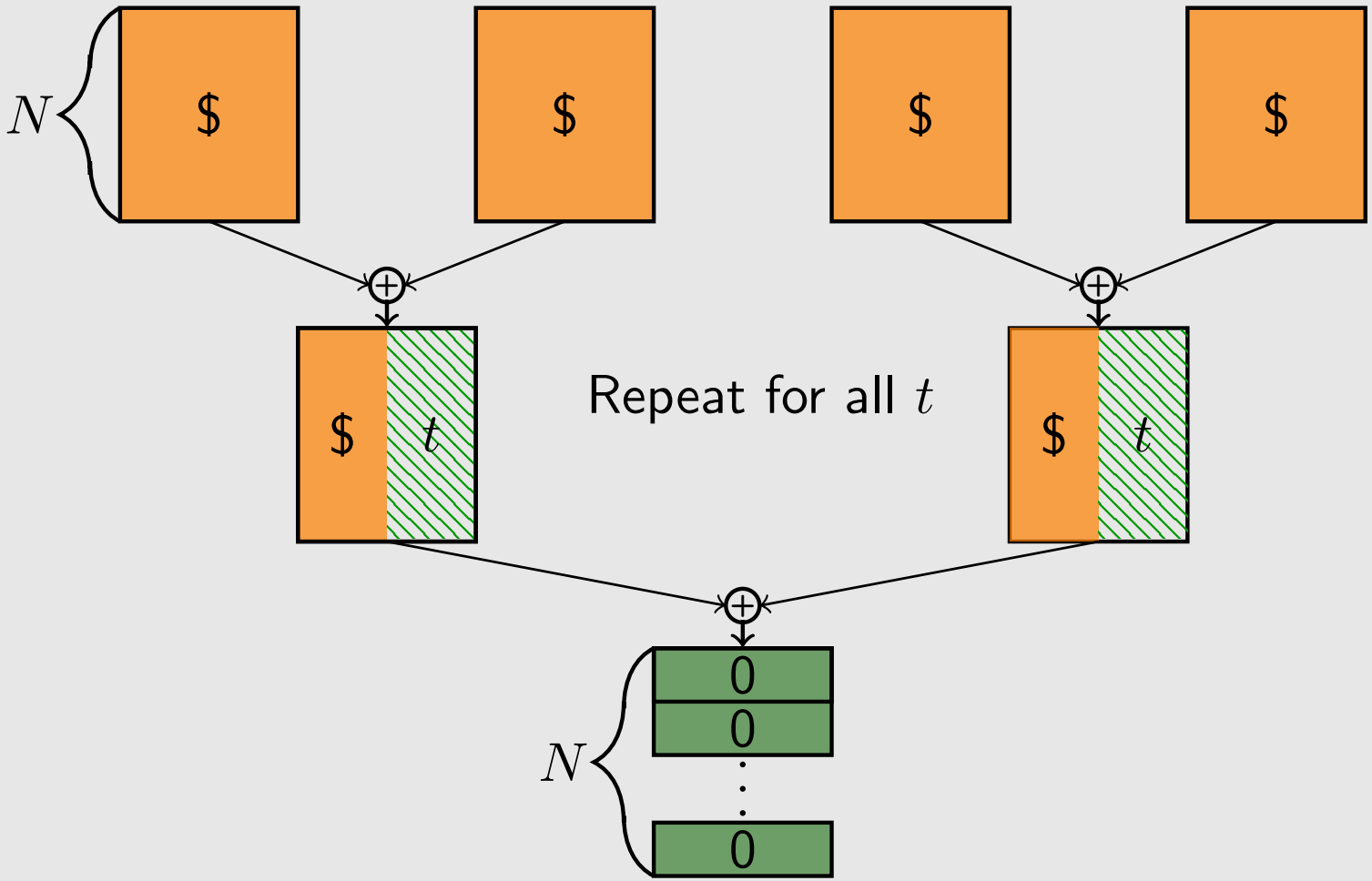

Better TMTO for BKW

Schroeppel-Shamir Theorem [SS81, HGJ10]

Schroeppel-Shamir solves the 4-sum Problem in time T, memory M:

$$T = N^2 \text{ and } M = N$$

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Dissection solves the c-sum Problem in time T and memory M:

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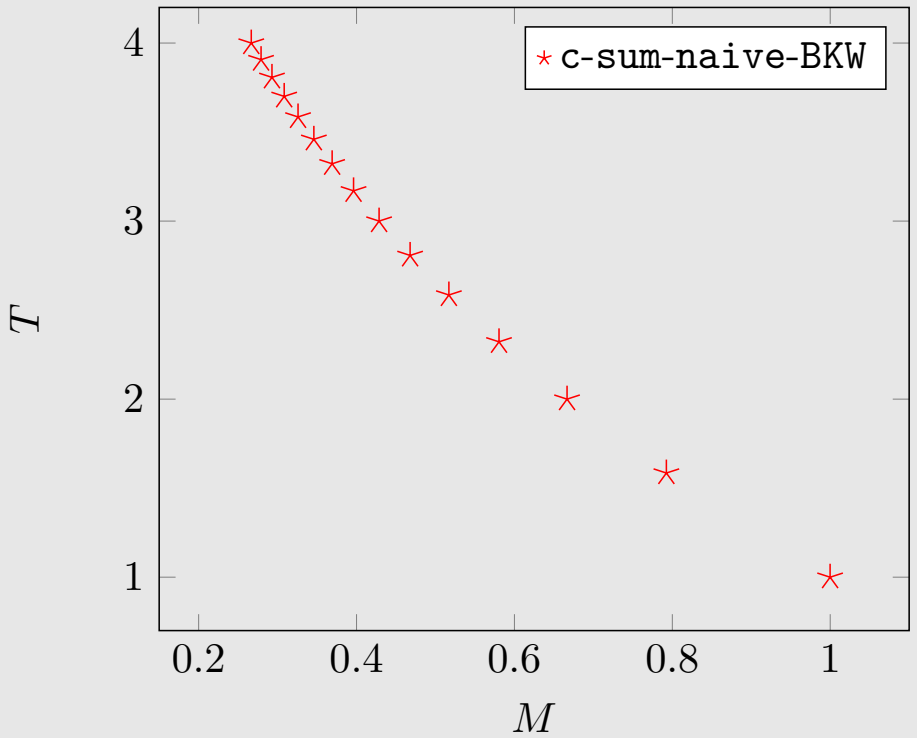

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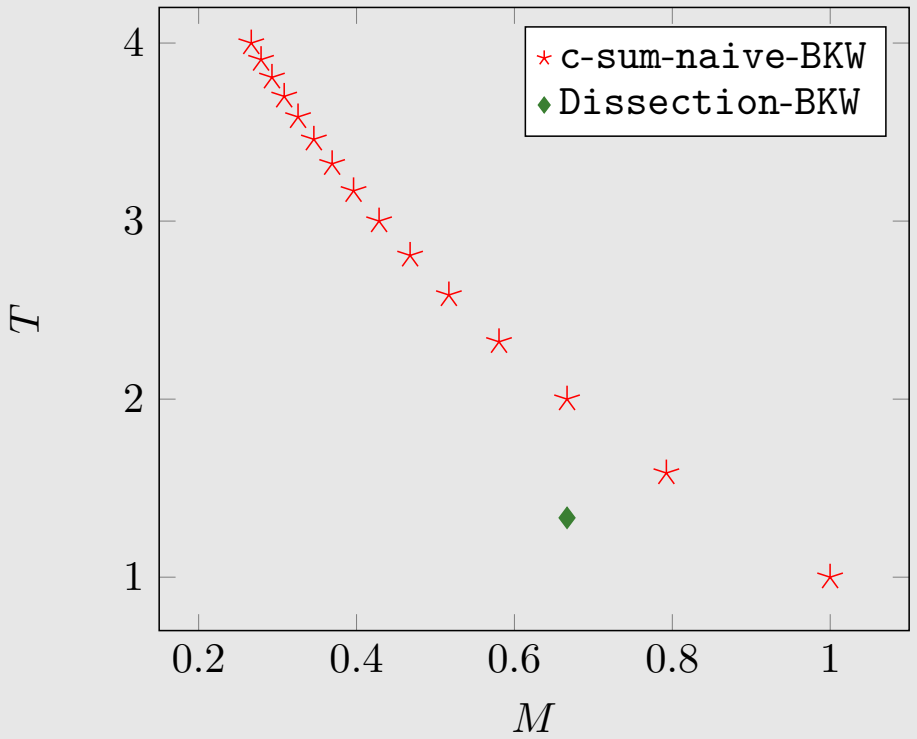

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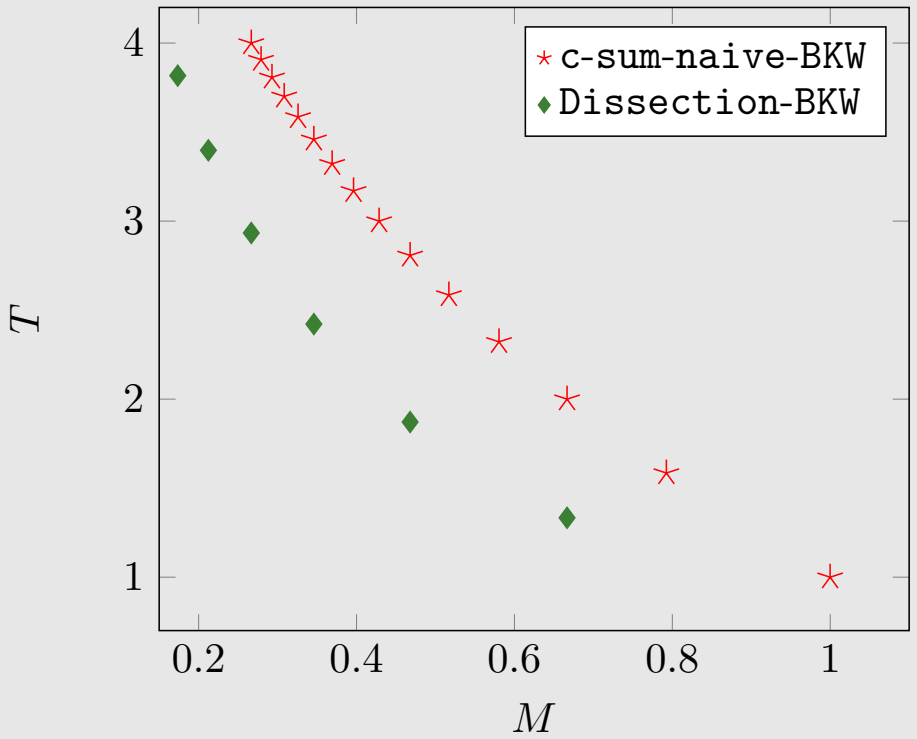

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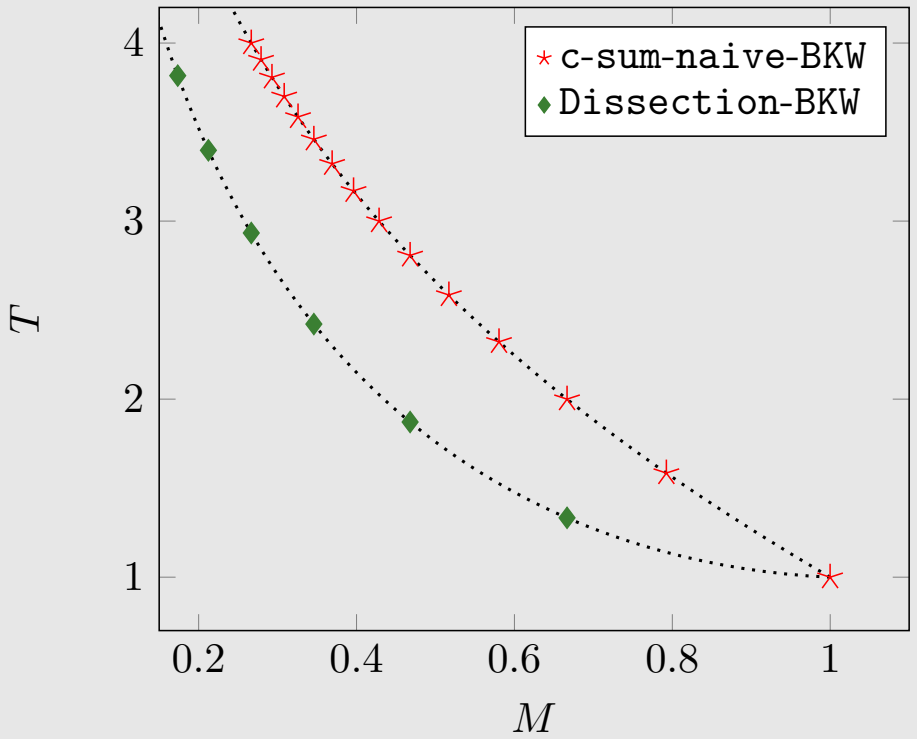

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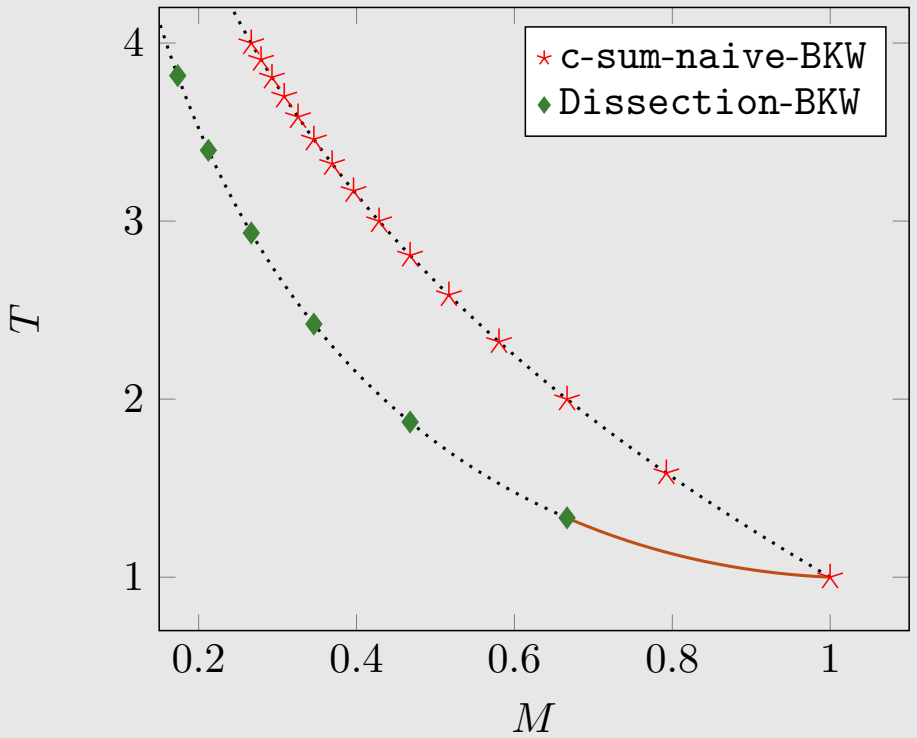
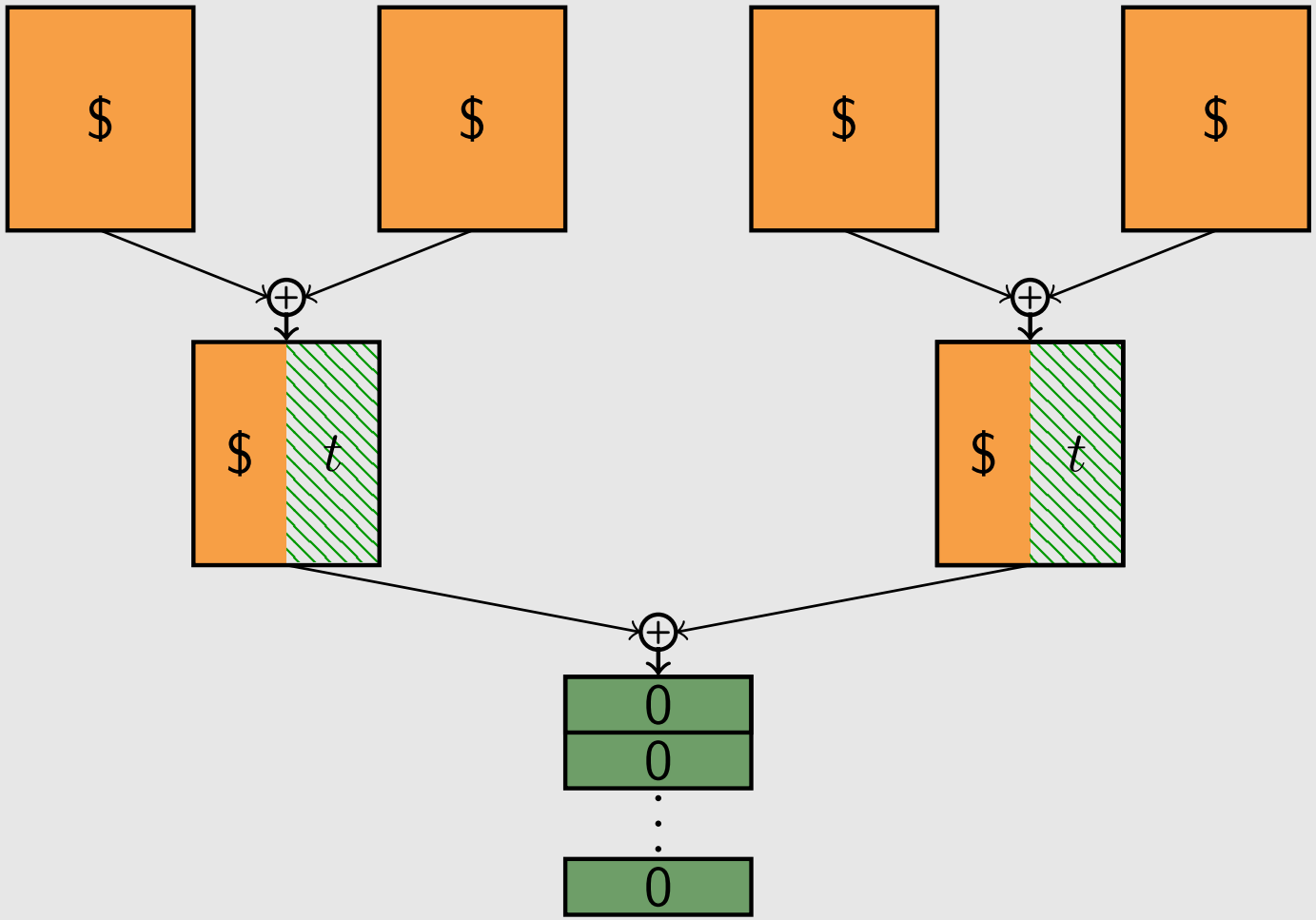


Tailored Schroeppel-Shamir



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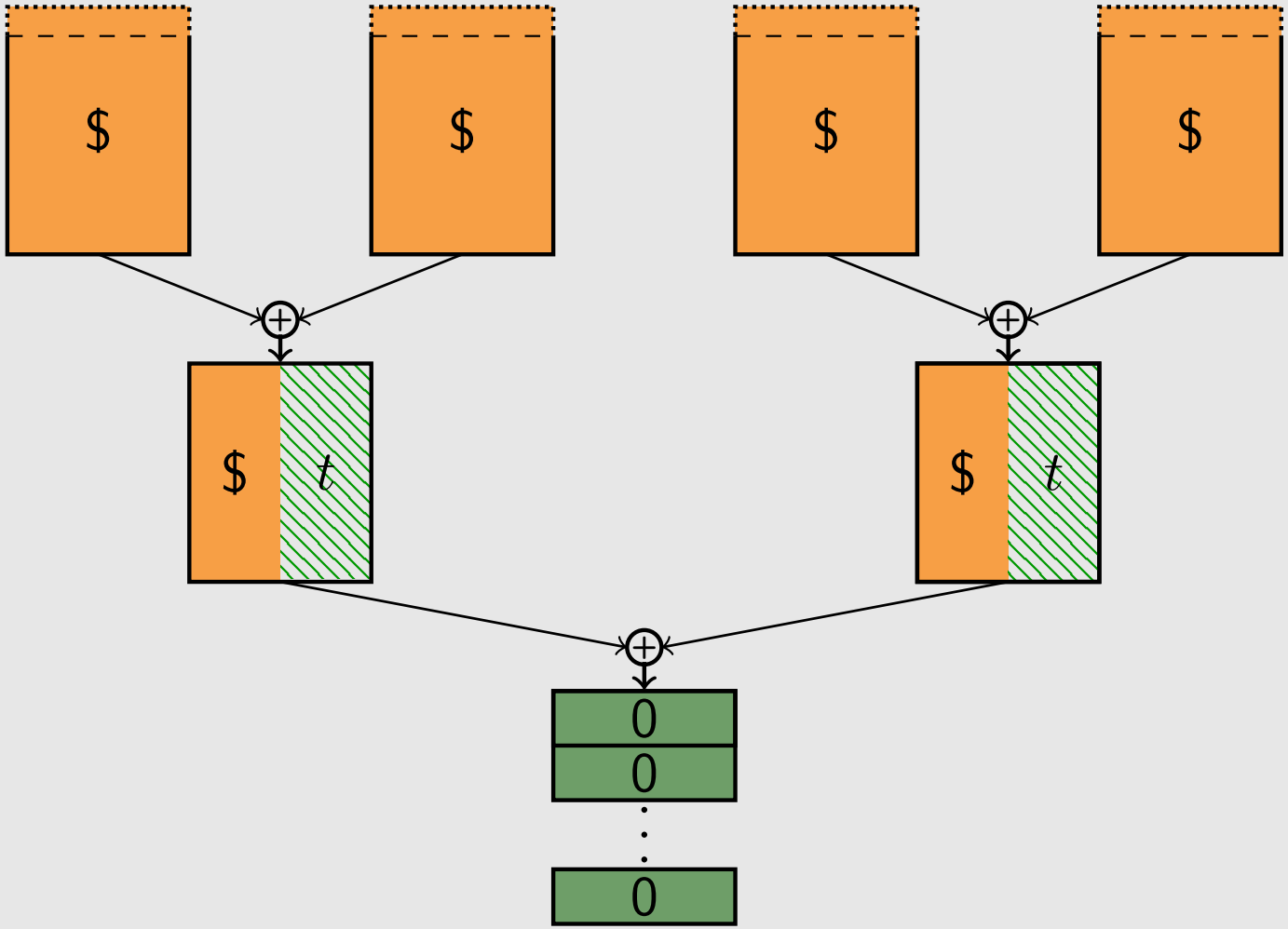

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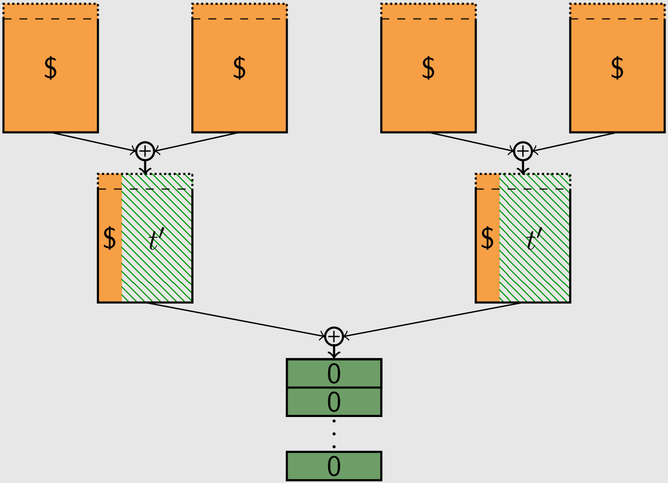

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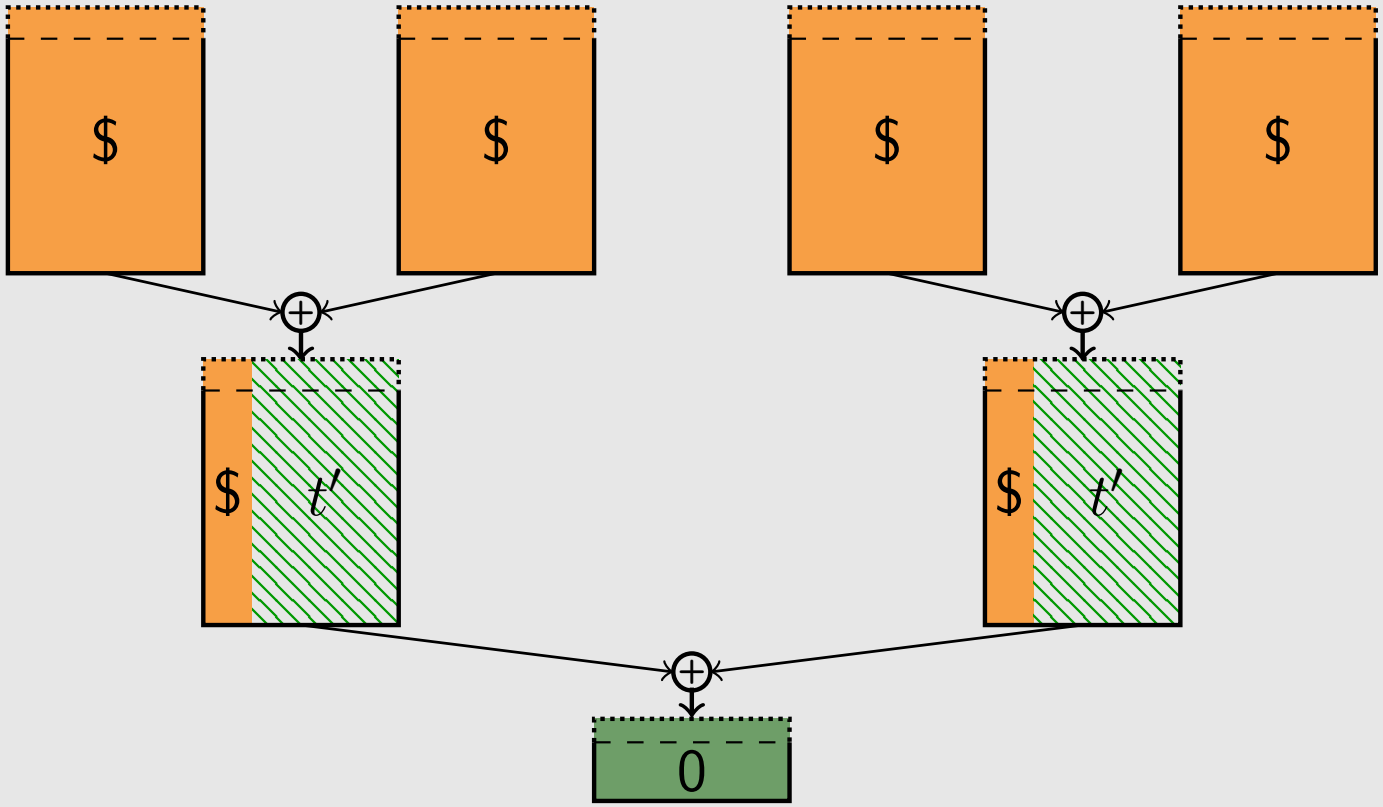

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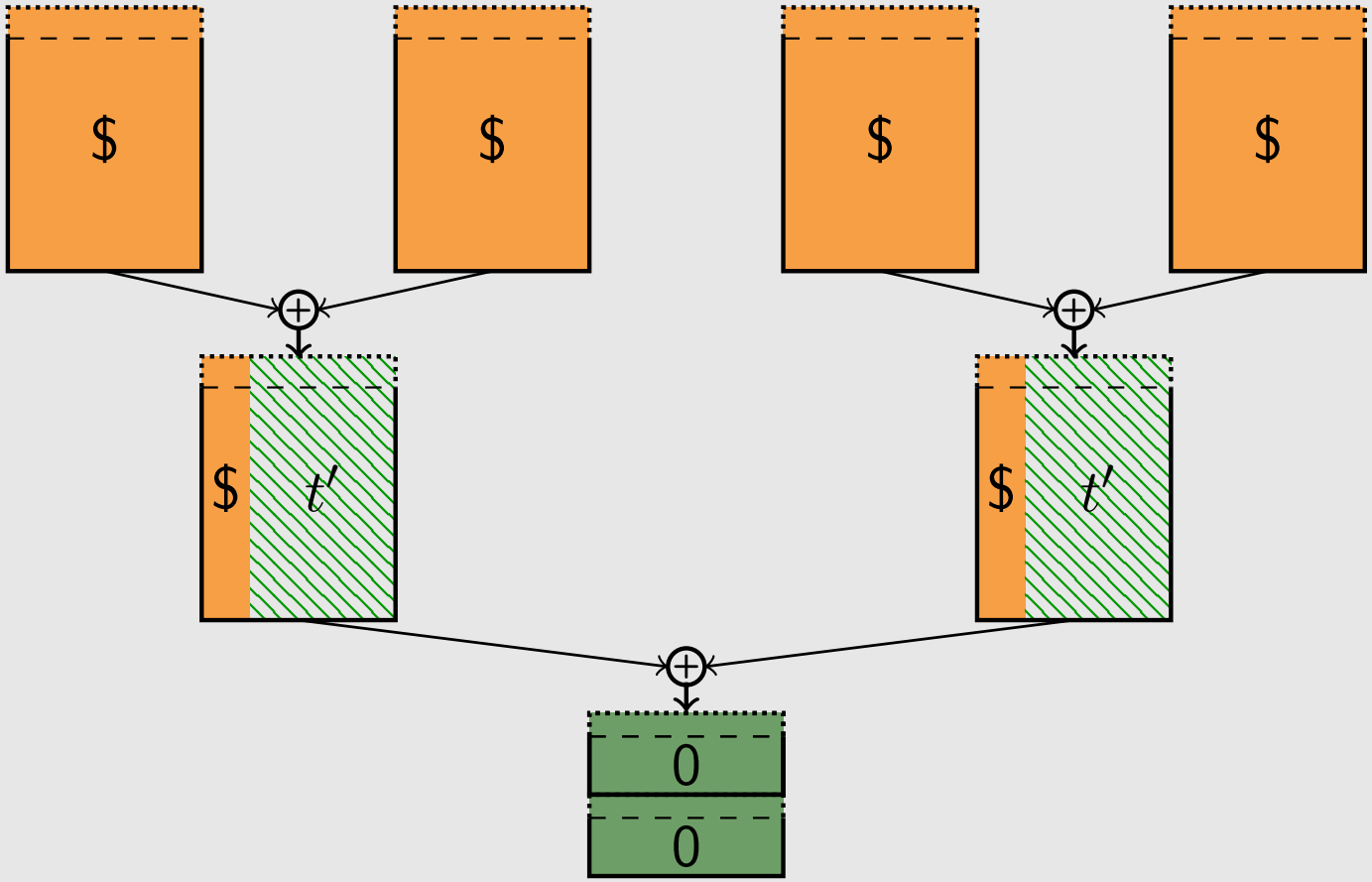

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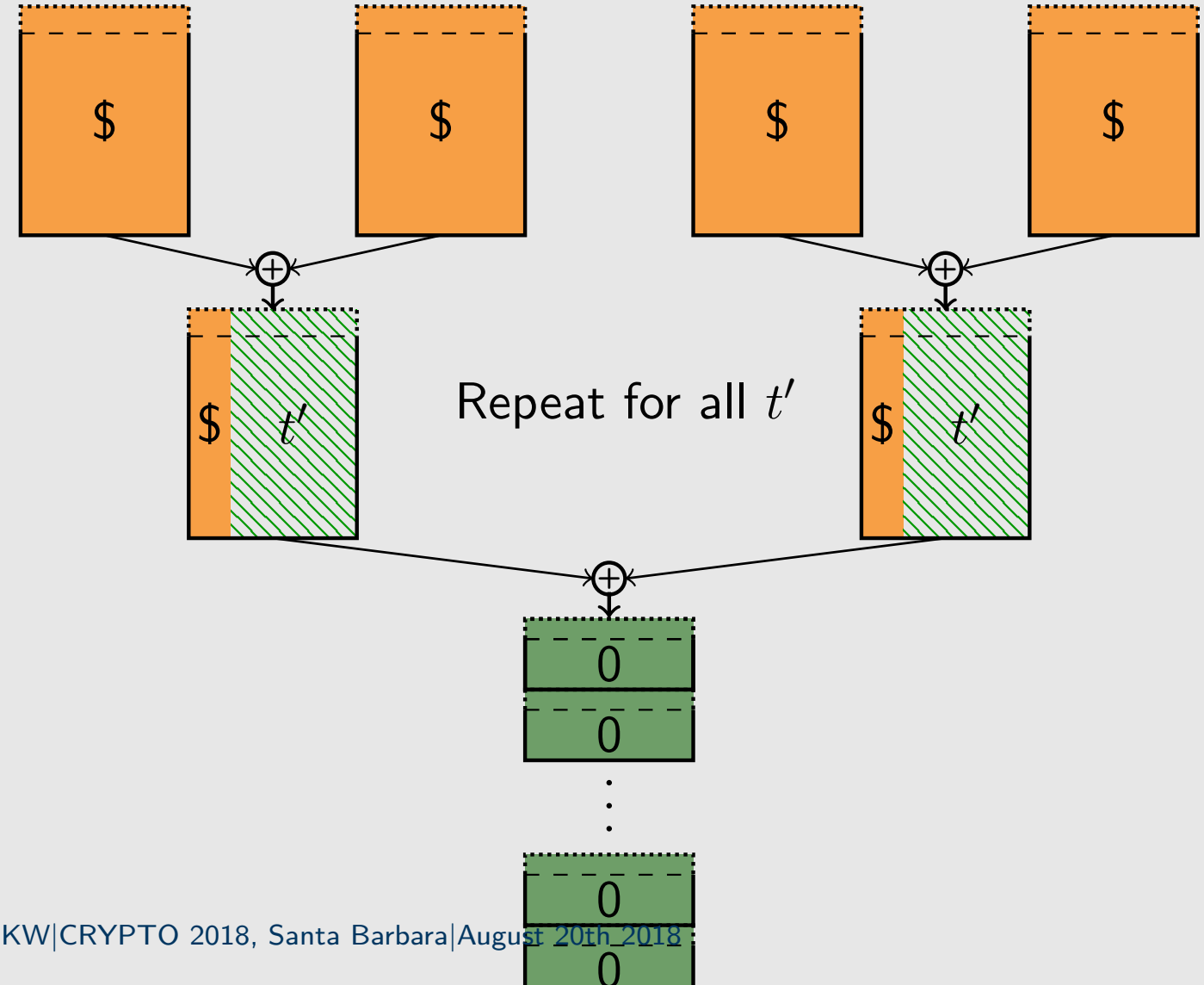

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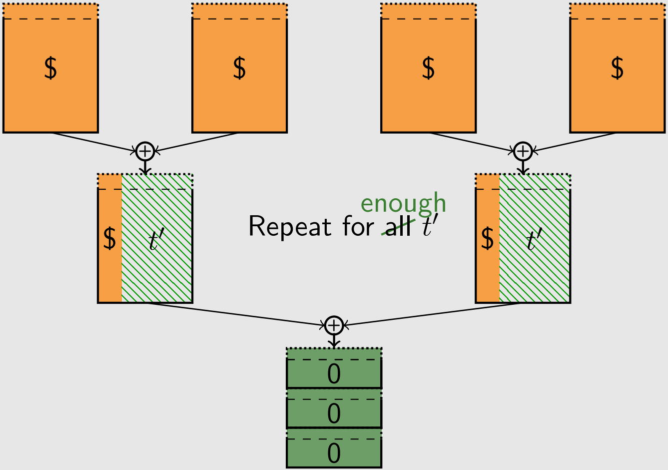

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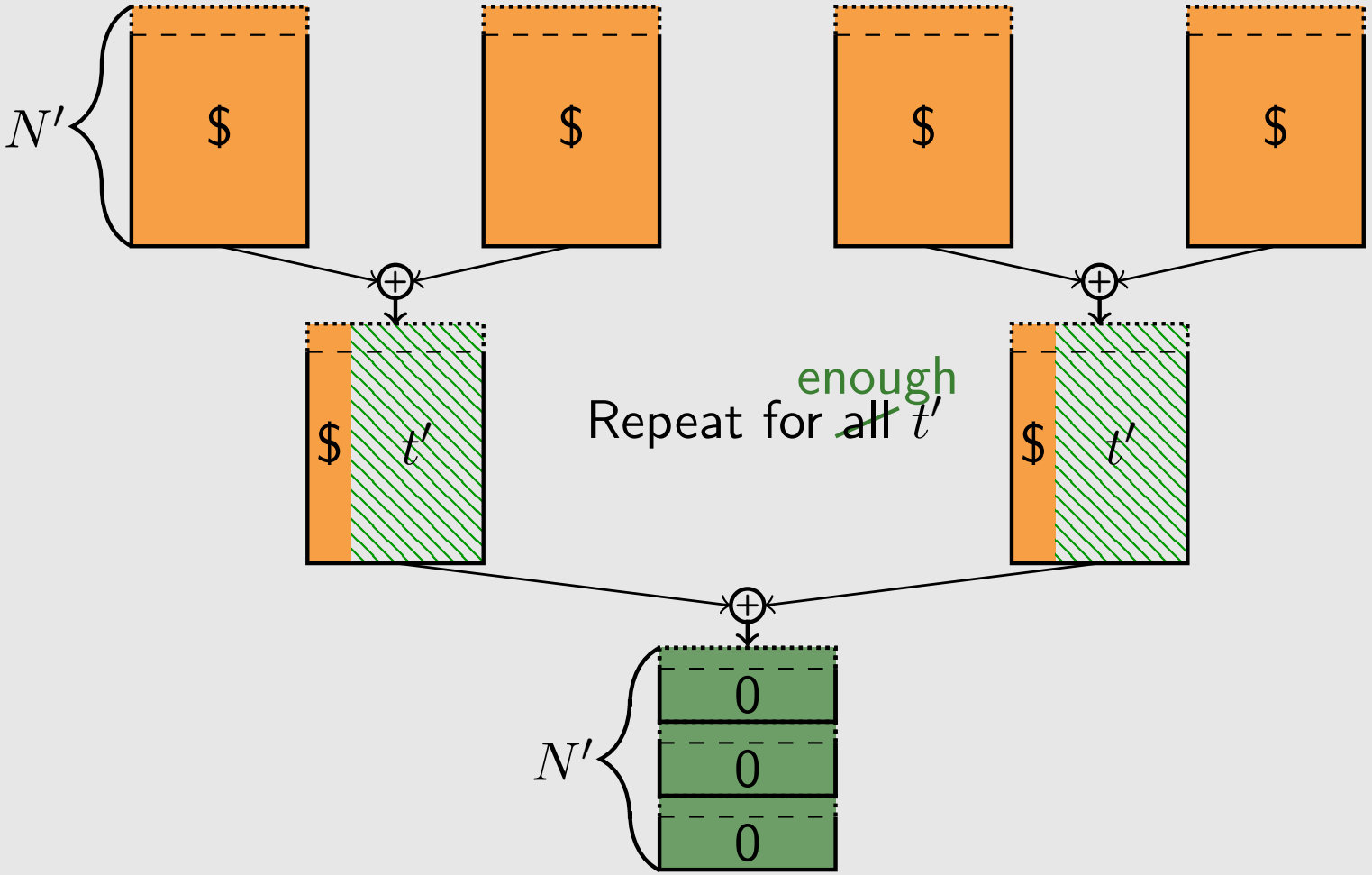
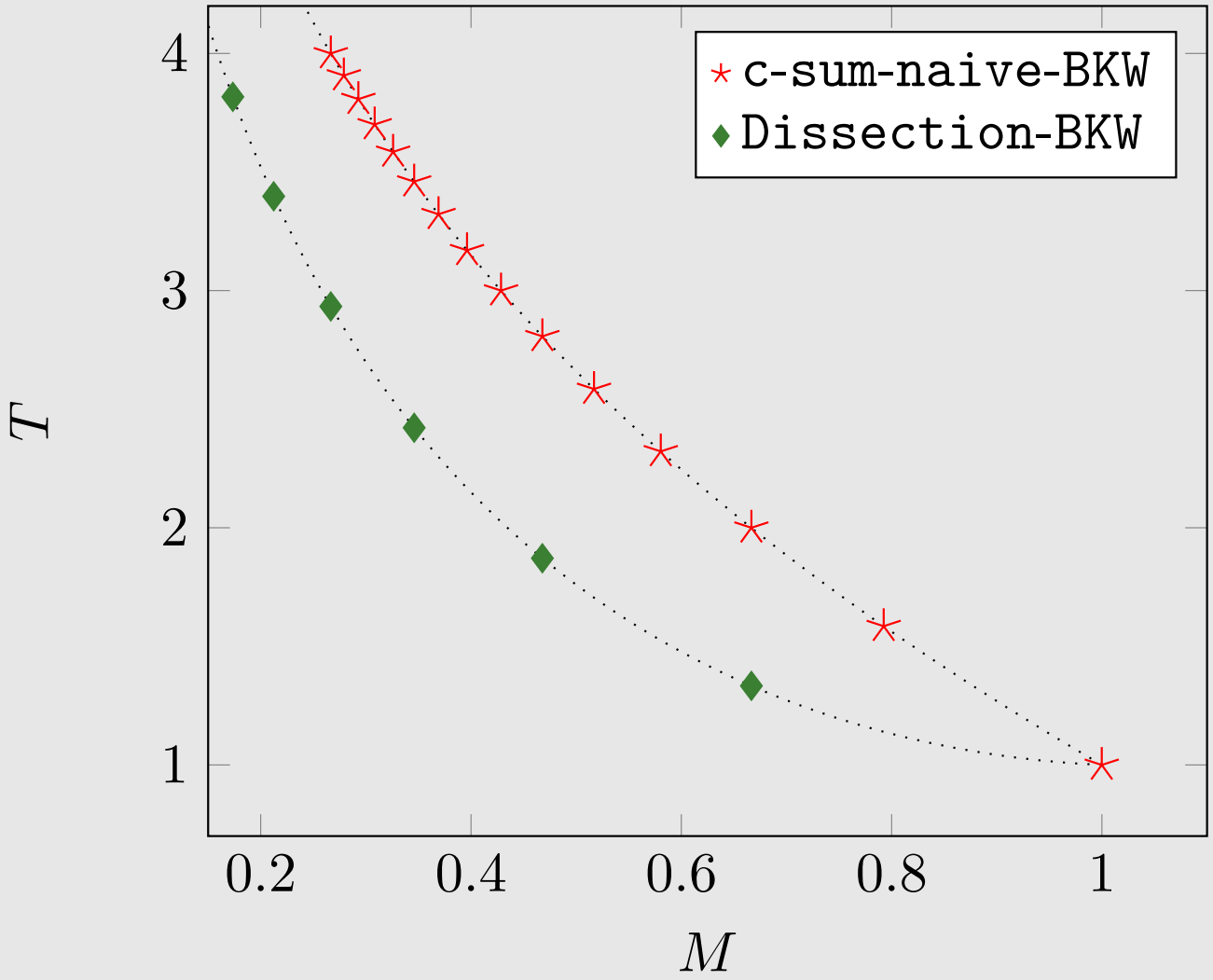


A continous TMTO via tailored Dissection



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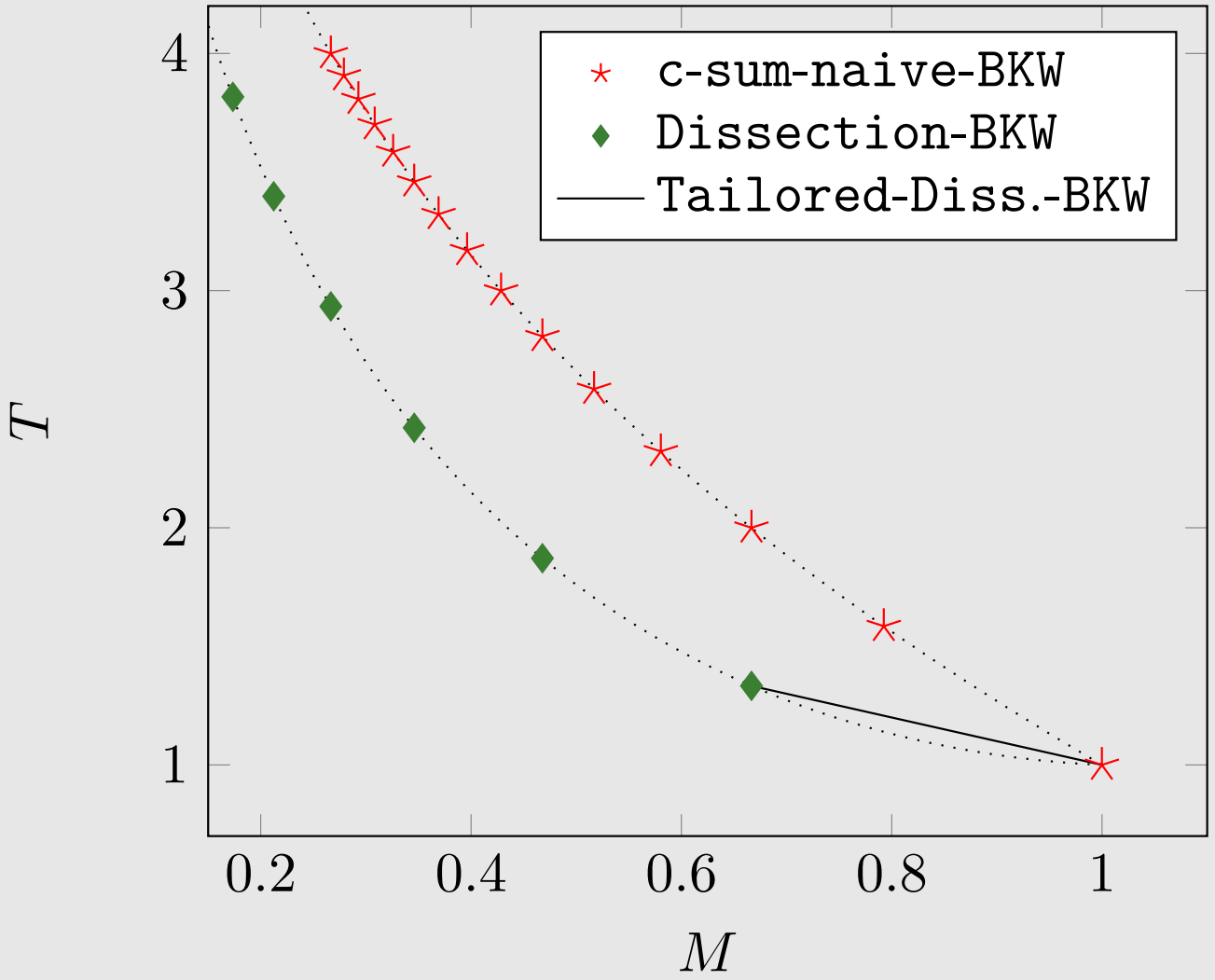

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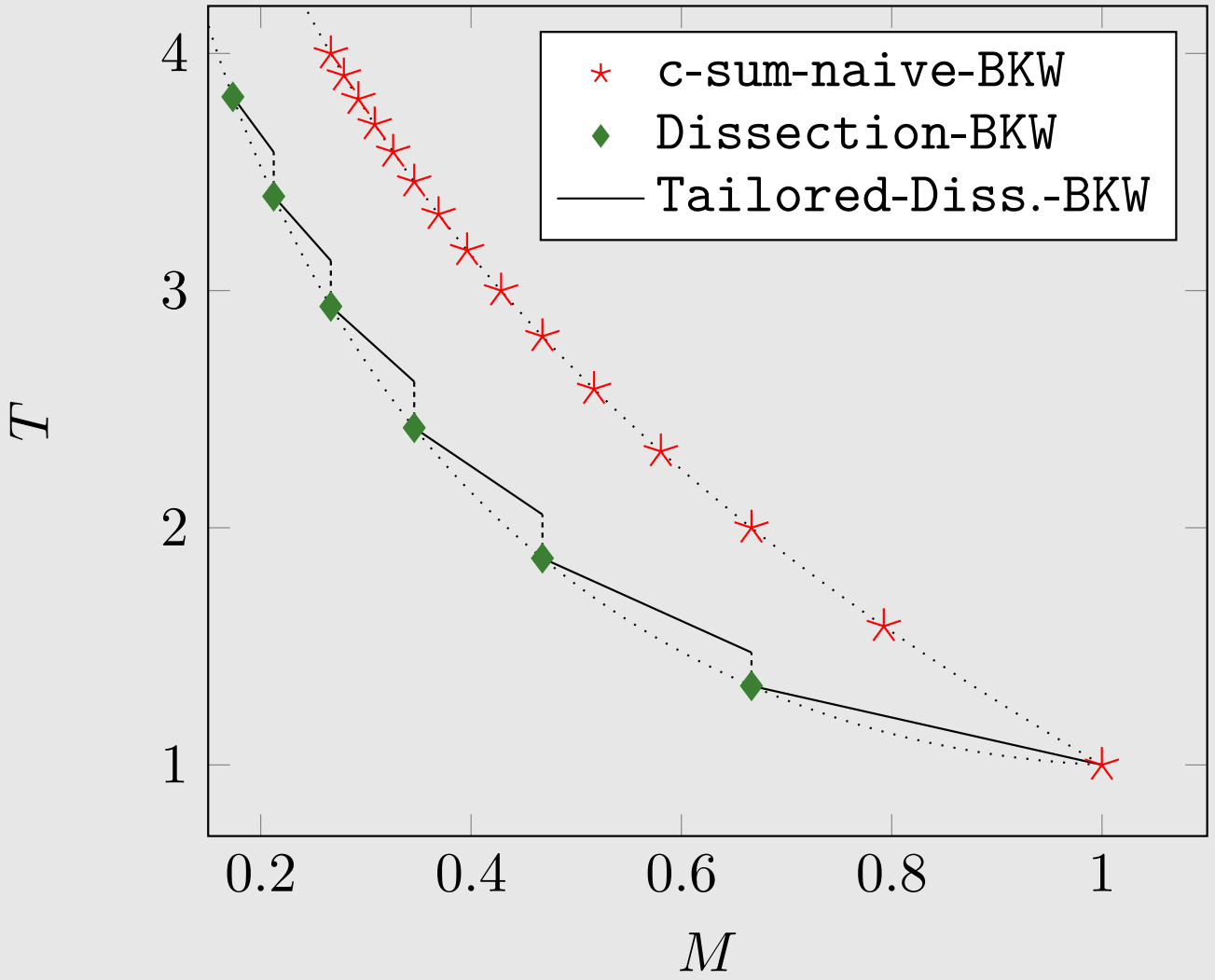

Results

- BKW-variant applicable for any given amount of memory

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