Trapdoor functions from the Computational Diffie-Hellman Assumption

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August 22, 2018

Classical Public-Key Crypto

TDF [DH76]

New Directions in Cryptography

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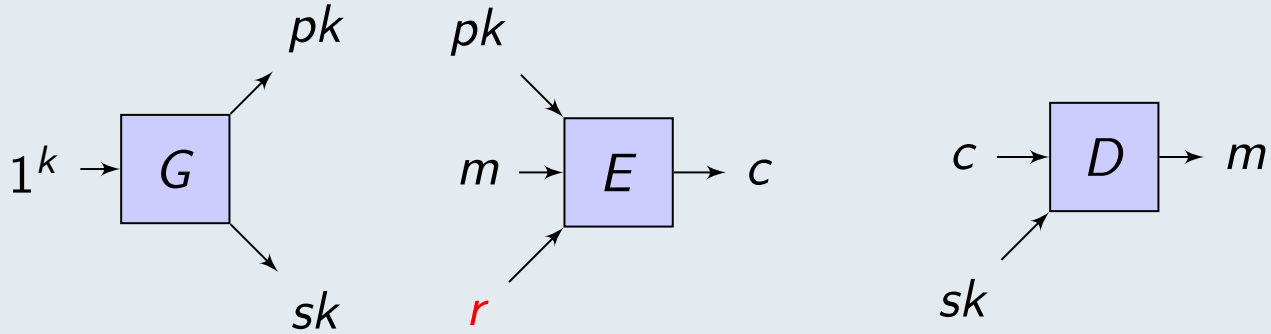
New Directions in Cryptography

PKE [GM82]

Probabilistic Encryption &
How To Play Mental Poker Keeping Secret
All Partial Information

PKE and TDF

PKE



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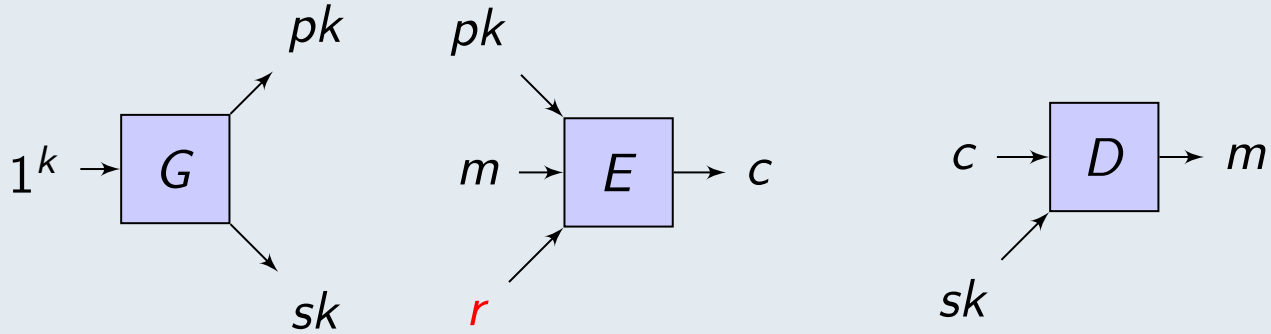

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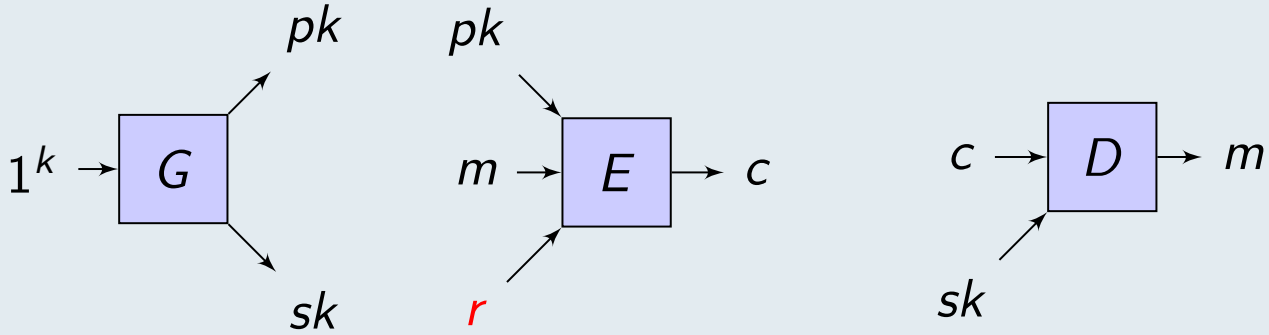

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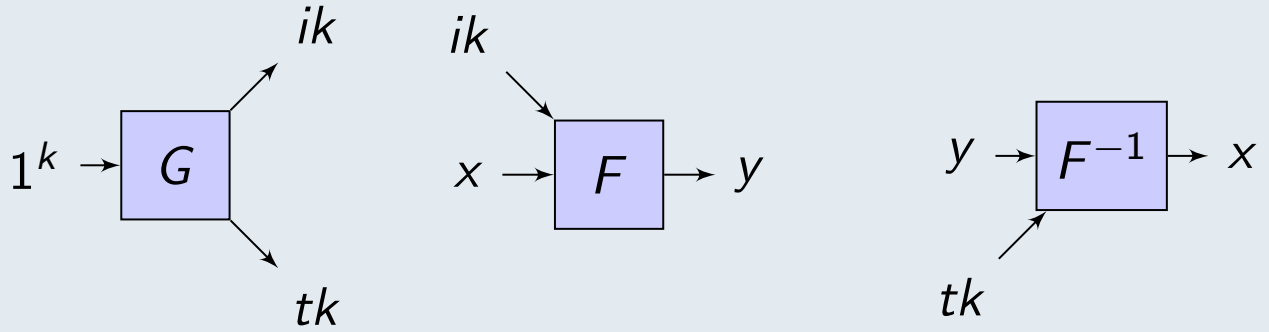

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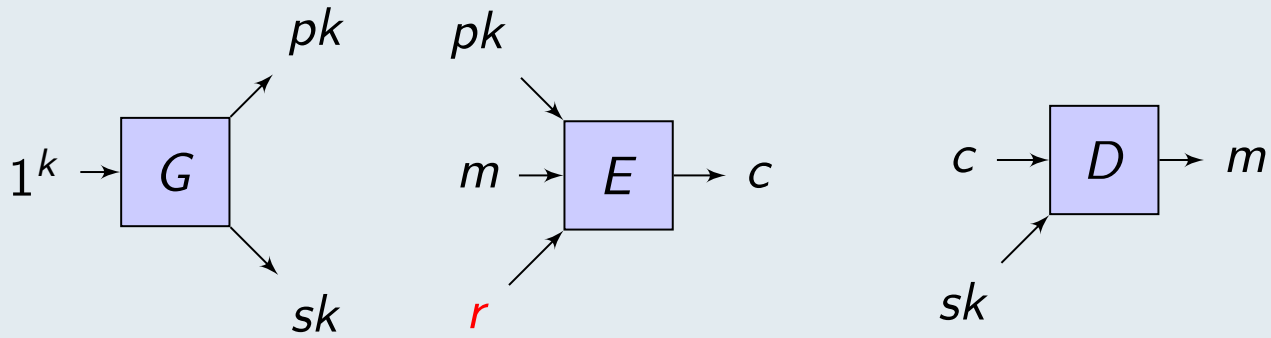

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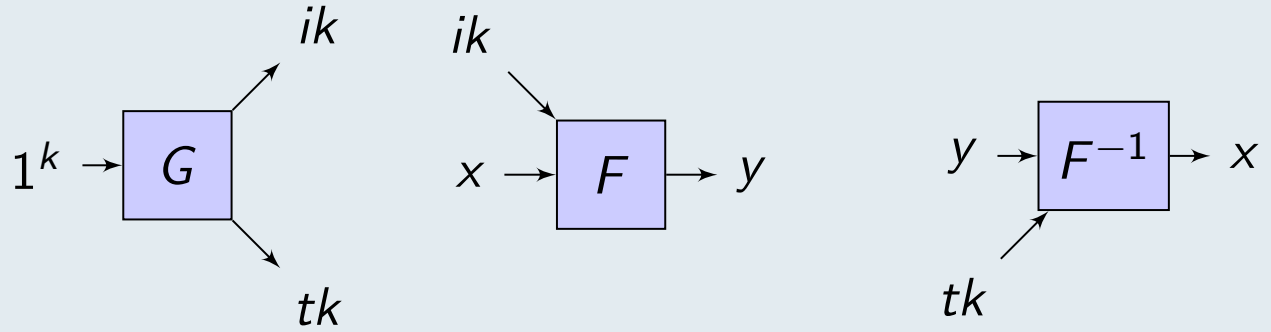

One-wayness Security: $(ik, \mathrm{F}(ik, x)) \stackrel{?}{\rightarrow} x$ is hard for random ik, x.

TDF vs PKE

Main Difference

- No randomness used in the evaluation algorithm of TDF.

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- TDF implies the existence of PKE. [Yao'82, GM'82].
- TDF impossible from PKE w.r.t. black-box techniques [GMR'01].

TDF Usefulness

ik_1, ik_2

ik_1, ik_2 and tk_1

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$$y_1 = F(ik_1, x_1), y_2 = F(ik_2, x_2)$$

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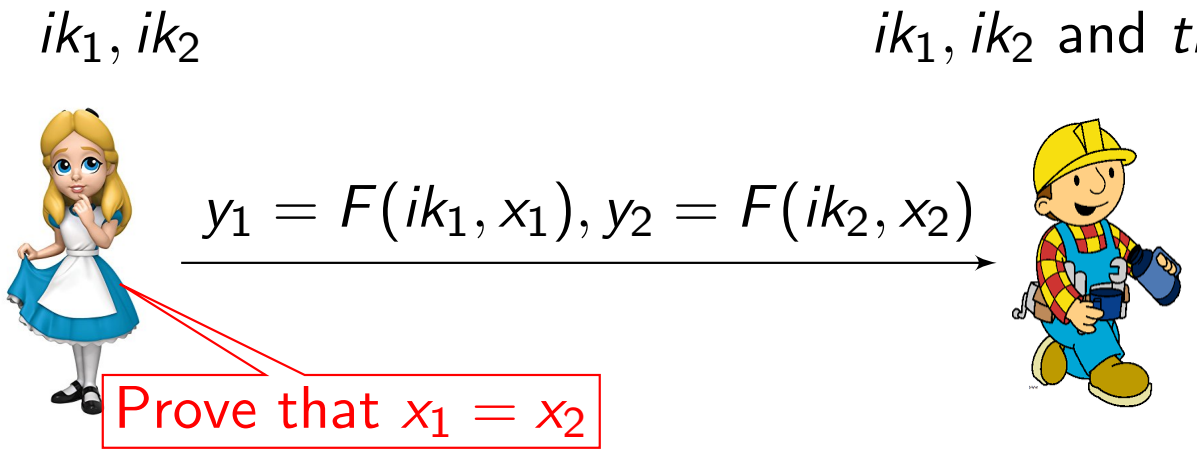

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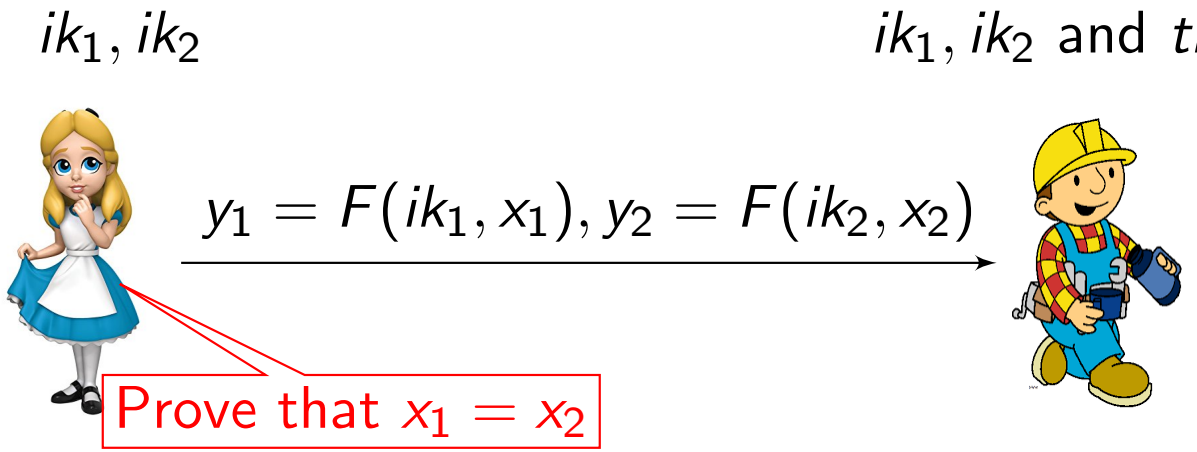

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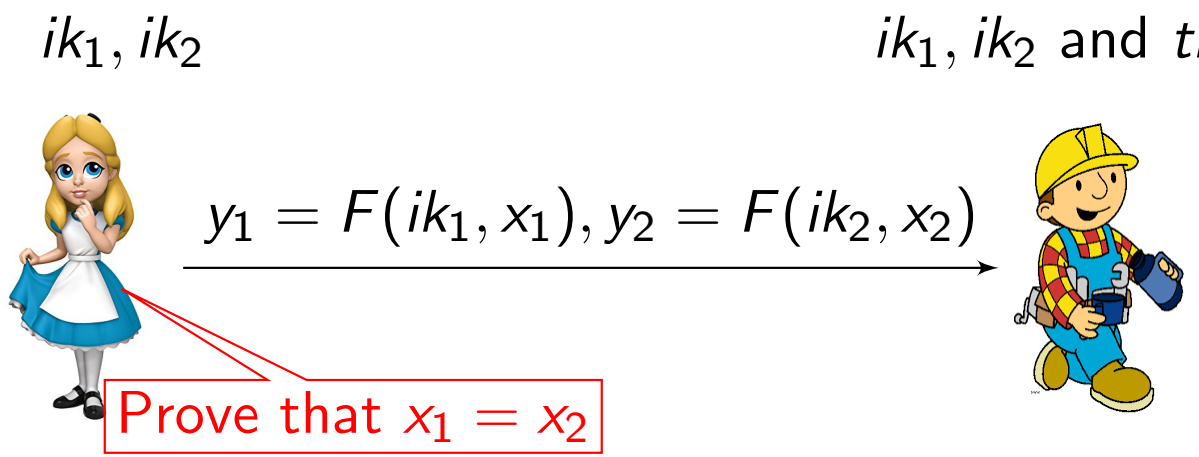

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PKE instead of TDF

- Consistency check: require some kind of proof (e.g., NIZK). [BFY90,NY90]

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This talk: We can do it from CDH.

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Decisional Diffie-Hellman (DDH)

- $(g, g^x, g^y, g^{xy}) \stackrel{c}{\equiv} (g, g^x, g^y, g^z)$, where $x, y, z \leftarrow \mathbb{Z}_p$

Why is CDH Preferable?

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- CDH is a weaker assumption.
  - There are groups in which CDH is conjectured to be hard but DDH is easy (e.g., $\mathbb{Z}_p^*$, groups with pairings).

Main Challenge in Building TDF from DH-Related Assumptions

Why is constructing TDF from Diffie-Hellman assumptions difficult?

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It doesn't naturally offer trapdoors!

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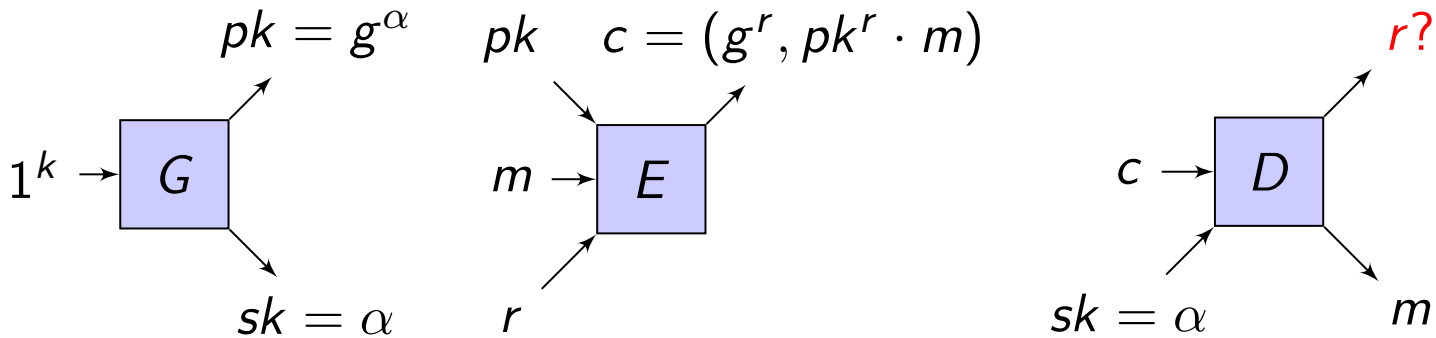

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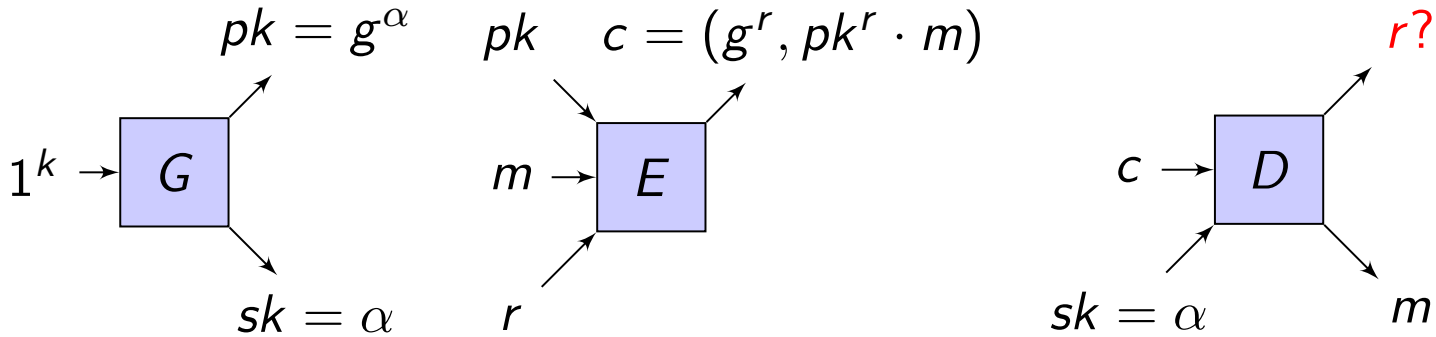

Main bottleneck in designing TDFs

- Recovering r: solving the Discrete Log!

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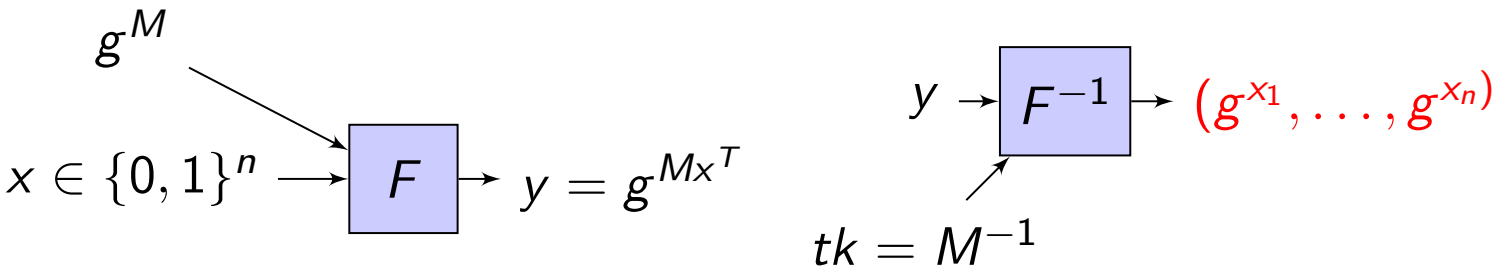

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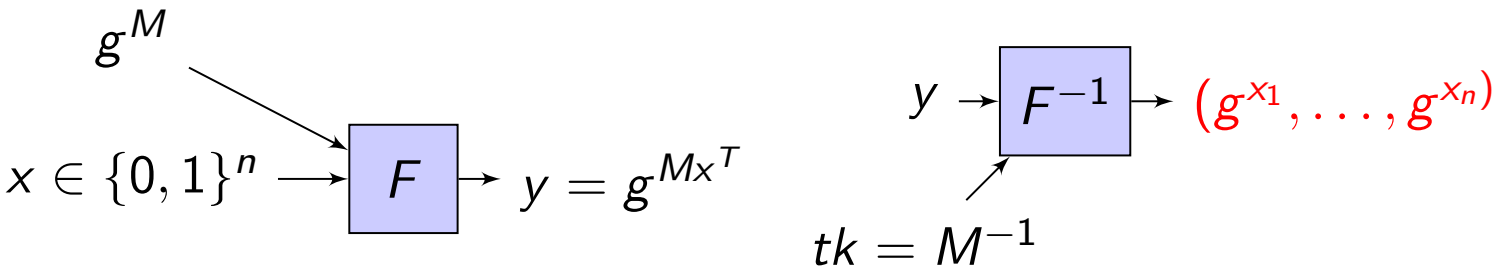

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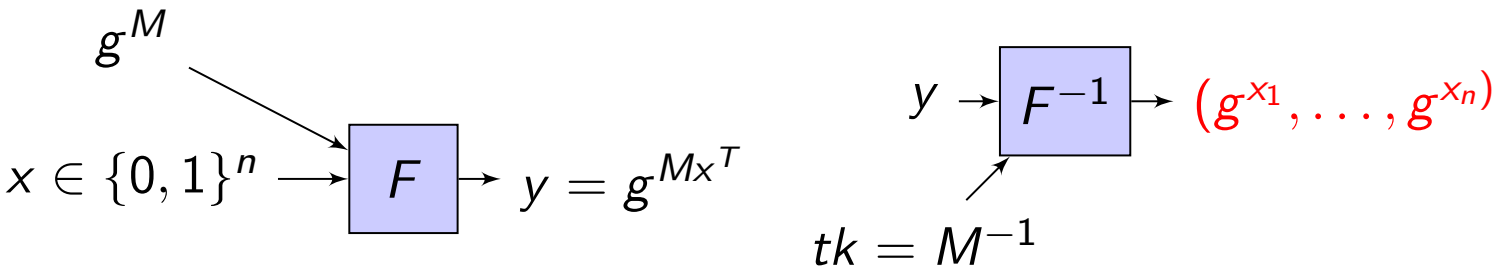

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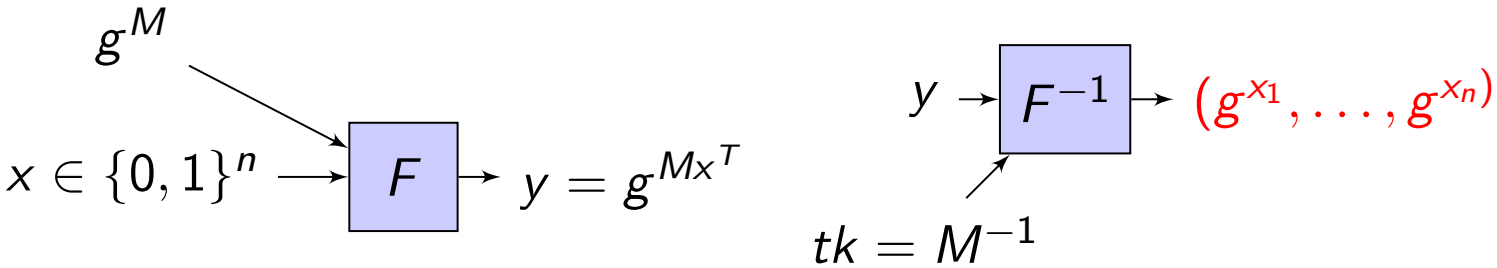

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- CDH is not known to imply rank indistinguishability.

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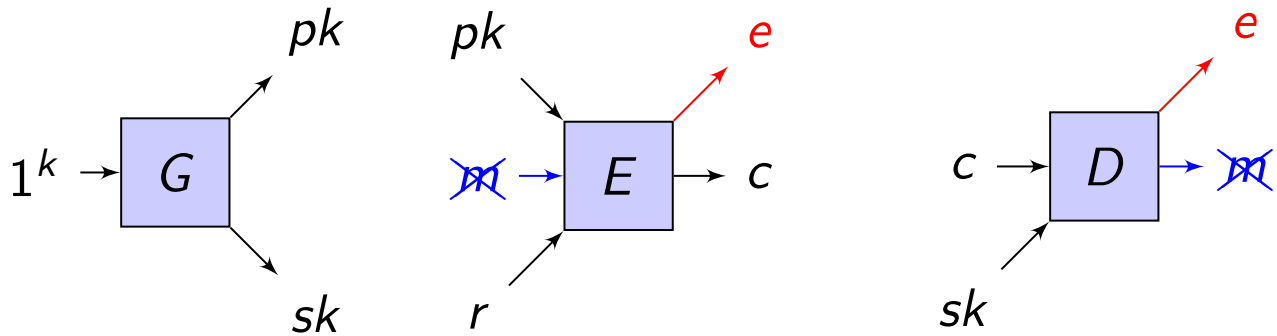
Plan for the Rest of the talk

- Define Recyclable Targeted KEM
- CDH $\Rightarrow$ Recyclable Targeted KEM (Not discussed. See [DG'17].)
- Recyclable Targeted KEM $\Rightarrow$ TDF

Key-Encapsulation Mechanism



e is always a single bit.

Recyclable Targetted KEM

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Targeting Property [DG'17]

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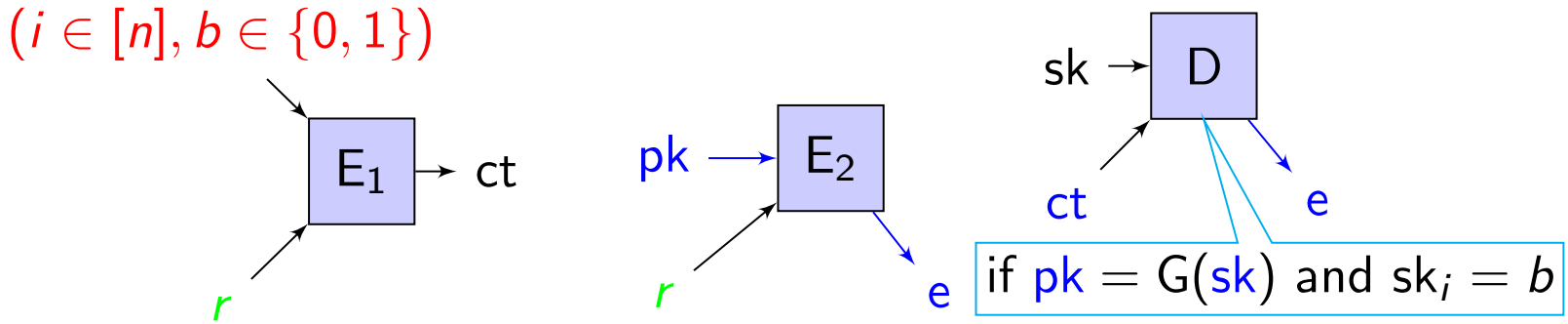
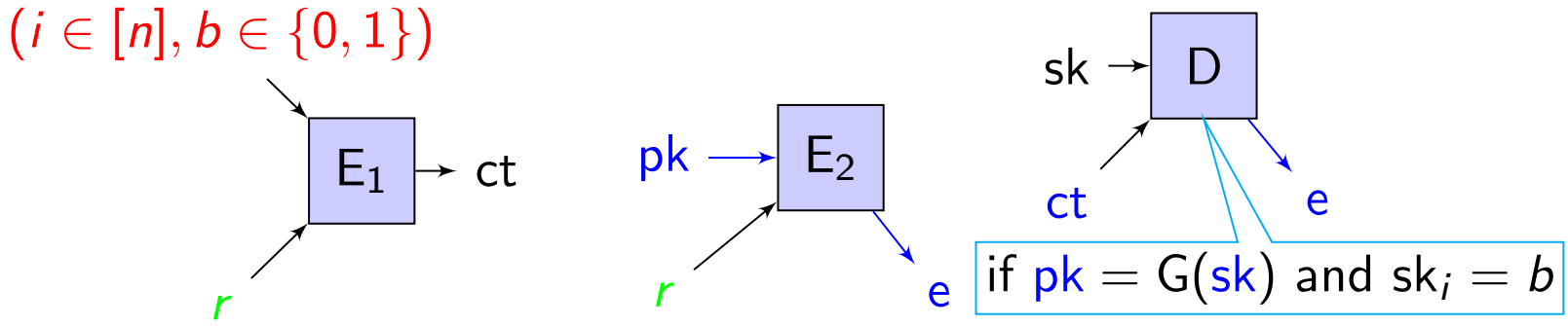
Recyclability

ct does not depend on pk. So
$\mathsf{E}(\mathsf{pk}, (i, b); r) = (\mathsf{E}_1((i, b); r), \mathsf{E}_2(\mathsf{pk}, (i, b); r)) = (\mathsf{ct}, e)$


$(i \in [n], b \in \{0,1\})$
E_1
ct
r
pk
E_2
r
e
sk
D
ct
e
if pk = G(sk) and $sk_i = b$



Simple construction for recovering the first bit of the input.

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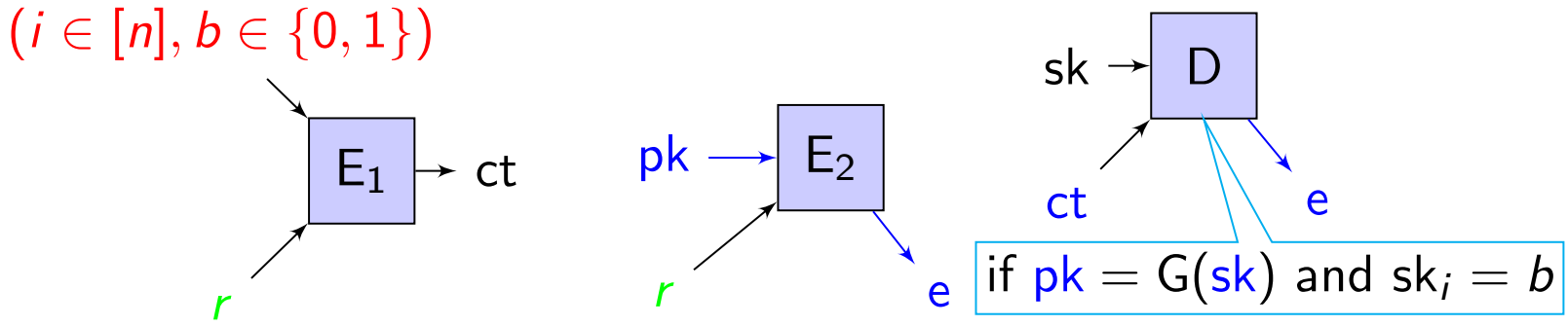

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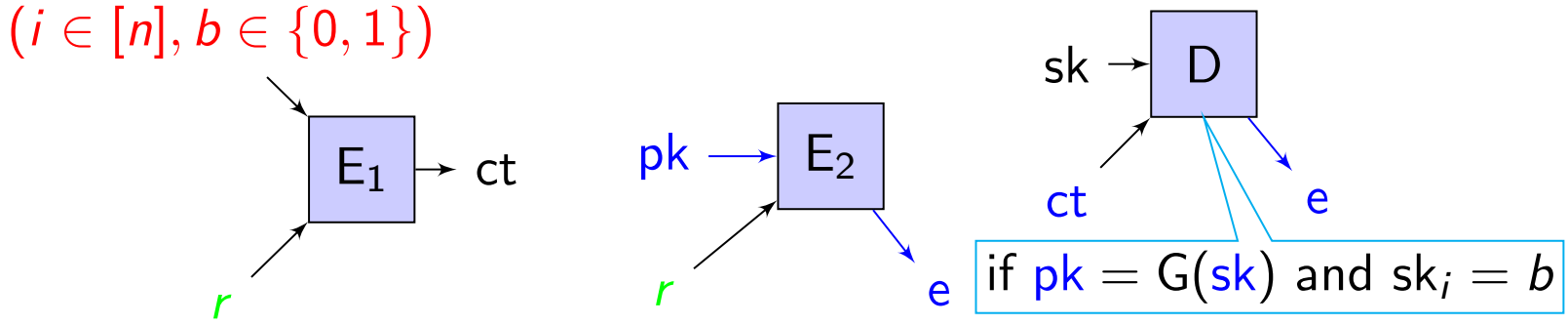

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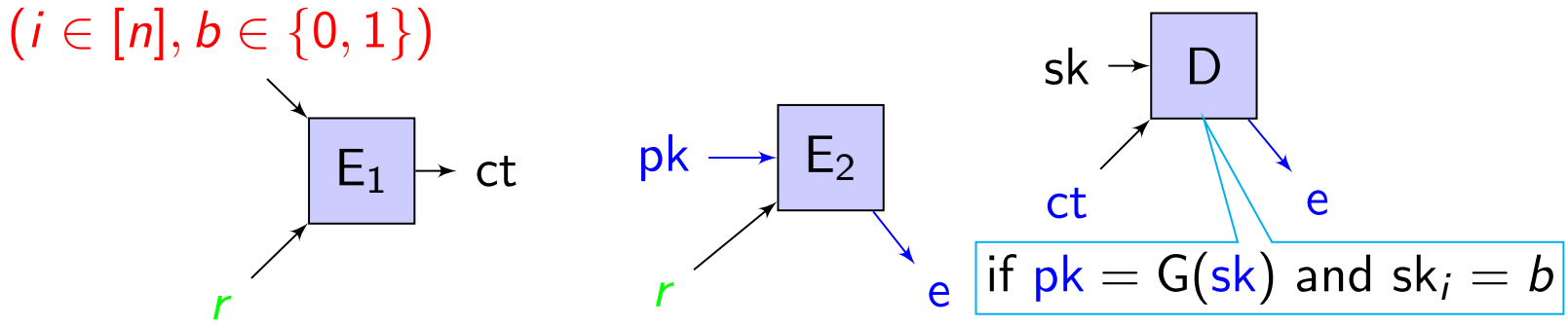

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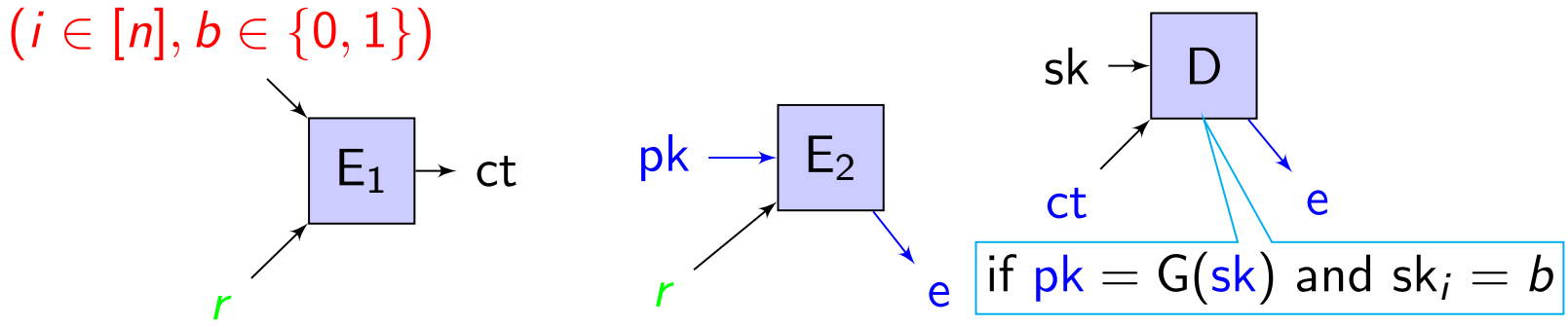

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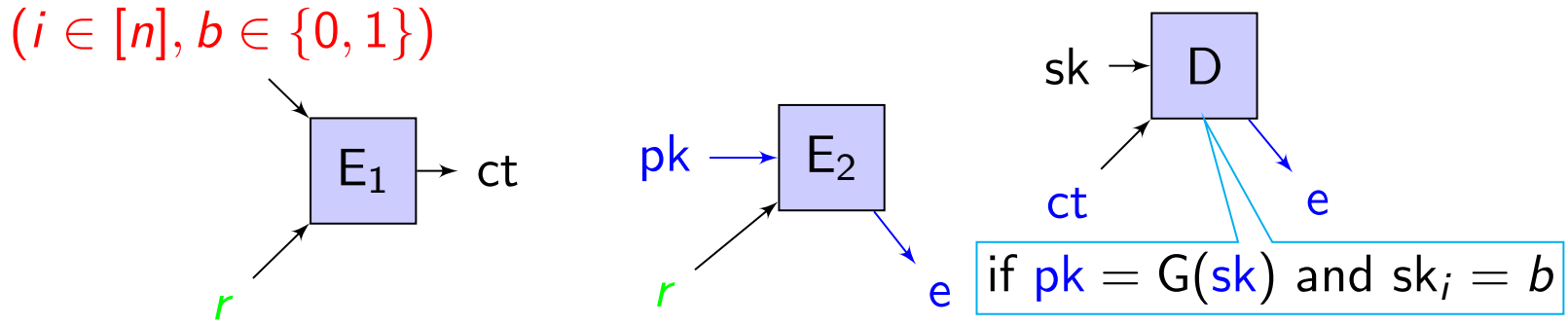

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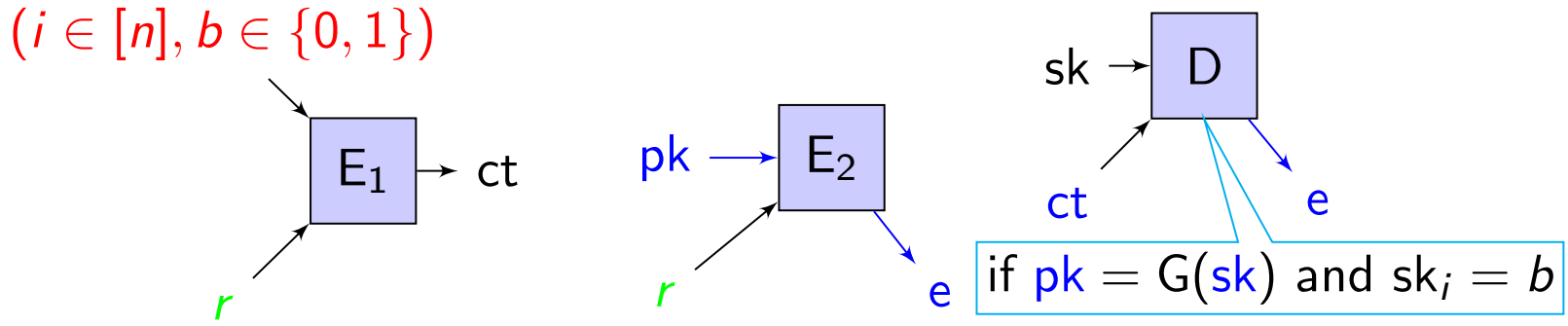

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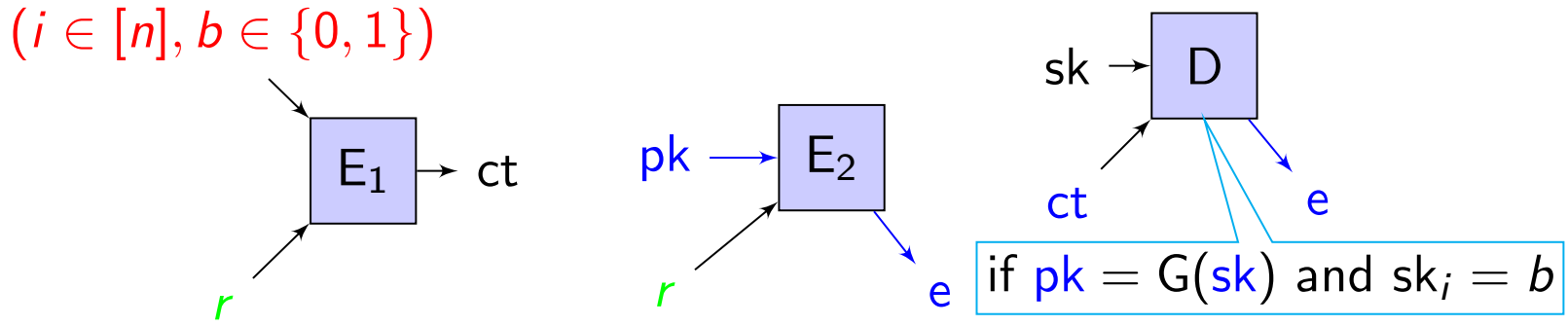

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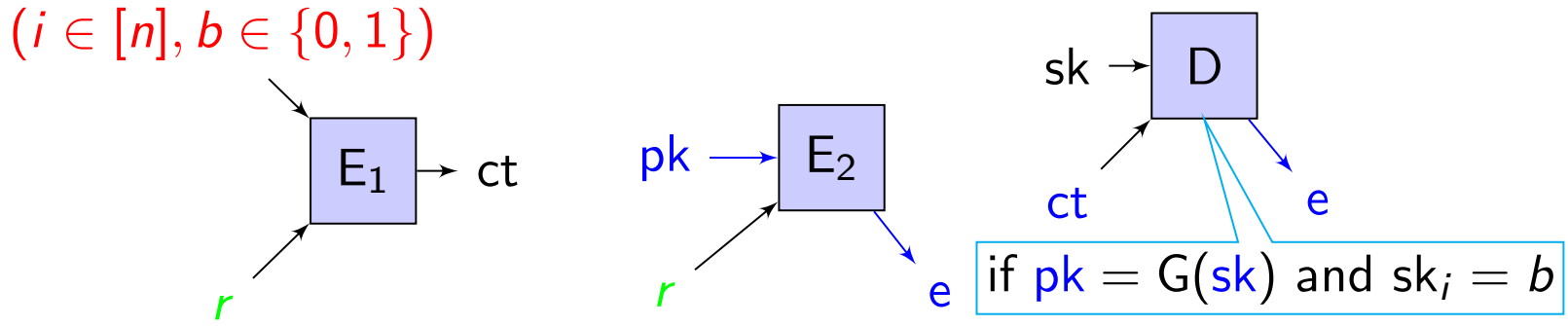
- Can recover sk_1 with probability $1/2$. This can be boosted via repetition.
- Not clear how to prove security!



Simple construction for recovering the first bit of the input.

- $\mathsf{tk} = \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$ and $\mathsf{ik} = \begin{pmatrix} \mathsf{ct}_1 \\ \mathsf{ct}_1' \end{pmatrix} = \begin{pmatrix} \mathsf{E}_1((i{=}1, b{=}0); r_1) \\ \mathsf{E}_1((i{=}1, b{=}1); r_1') \end{pmatrix}$
- $\mathsf{F}(\mathsf{ik}, \mathsf{sk})$: let $\mathsf{pk} = \mathsf{G}(\mathsf{sk})$.
  - if $\mathsf{sk}_1 = 0$, then return $(\mathsf{pk}, \mathsf{D}(\mathsf{sk}, \mathsf{ct}_1)) = (\mathsf{pk}, \mathsf{E}_2(\mathsf{pk}; r_1))$.
  - if $\mathsf{sk}_1 = 1$, then return $(\mathsf{pk}, \mathsf{D}(\mathsf{sk}, \mathsf{ct}_1')) = (\mathsf{pk}, \mathsf{E}_2(\mathsf{pk}; r_1'))$.
- F^{-1}: Check for a match:

$$\begin{pmatrix} \mathsf{E}_2(\mathsf{pk}; r_1) \\ \mathsf{E}_2(\mathsf{pk}; r_1') \end{pmatrix}$$

- Can recover sk_1 with probability $1/2$. This can be boosted via repetition.
- Not clear how to prove security!
  - Fix: Put a random bit in the place you cannot apply D.

Recovering the First Bit

- $\mathsf{Gen}(1^\lambda)$: $\mathsf{tk} = \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$ and

$$\mathsf{ik} = \begin{pmatrix} \mathsf{ct}_1 \\ \mathsf{ct}_1' \end{pmatrix} = \begin{pmatrix} \mathsf{E}_1((i{=}1, b{=}0); r_1) \\ \mathsf{E}_1((i{=}1, b{=}1); r_1') \end{pmatrix}$$

- $\mathsf{F}(\mathsf{ik}, \mathsf{sk} || b_1)$: let $\mathsf{pk} = \mathsf{G}(\mathsf{sk})$. Then:
  - if $\mathsf{sk}_1 = 0$ then $\mathsf{M}_1 := \begin{pmatrix} \mathsf{D}(\mathsf{sk}, \mathsf{ct}_1) \\ \mathsf{b}_1 \end{pmatrix}$
  - if $\mathsf{sk}_1 = 1$ then $\mathsf{M}_1 := \begin{pmatrix} \mathsf{b}_1 \\ \mathsf{D}(\mathsf{sk}, \mathsf{ct}_1') \end{pmatrix}$

  Return $\mathsf{Y} = (\mathsf{pk}, \mathsf{M}_1)$.
- $\mathsf{F}^{-1}(\mathsf{tk}, \mathsf{Y})$:

$$\mathsf{M}_1' = \begin{pmatrix} \mathsf{E}_2(\mathsf{pk}, (1, 0); r_1) \\ \mathsf{E}_2(\mathsf{pk}, (1, 1); r_1') \end{pmatrix}$$

Summary and Future Work

Summary

- A Construction of TDFs from CDH.

Future Work

- Extended forms of TDFs from CDH (e.g., lossy trapdoor functions)
- Trapdoor Permutations from CDH/DDH?